J. Sci. Tarbiat Moallem University Vol. 11, No. 2
(Autumn 2012)

# Quasi- Secondary Submodules

A. J. Taherizadeh: Kharazmi University

## Abstract

Let $R$ be a commutative ring with non-zero identity and $M$ be a unitary R-module. Then the concept of quasi-secondary submodules of $M$ is introduced and some results concerning this class of submodules is obtained.

## 1. Introduction

Throughout this paper all rings are commutative with non-zero identity and all modules are unitary. In [4] L.Fuchs introduced and studied the concept of quasi-primary ideals (see also [5]). An ideal $I$ of a ring $R$ is called a *quasi-primary* ideal of $R$ if the radical of I is a prime ideal of $R$. This concept then generalized to modules, i.e., the concept of quasi-primary submodules of a module introduced and developed in [3]. Here, we introduce the dual notation, that is, the quasi-secondary submodules of a module and obtain some results concerning this class of submodules. In section 2, we obtain some preliminary properties of quasi-secondary submodules. Section 3 is devoted to the quasi-secondary submodules of a multiplication module. Now we define some concepts which will be needed in sequel.

Let $M$ be an R-module and $N$ a submodule of it. The ideal $\{r \in R \mid rM \subseteq N\}$ will be denoted by $(N :_R M)$; in particular $(0 :_R M)$ is called the annihilator of $M$. A non-zero submodule $N$ of $M$ is called a *secondary* (*resp.second*) submodule of $M$ if for each $r \in R$ the homothety $N \overset{r}{\to} N$ is surjective or nilpoten (resp. surjective or zero). In this case $\sqrt{(0 :_R N)}$ is a prime ideal, say $p$, and we call $N$ a $p$-*secondary* (resp.*a* $p$*- second*) submodule of $M$. We refer readers for more details concerning secondary (resp.second) submodulse to [9] (resp. [12] ).

**KeyWords:** quasi – secondary submodules, secondary submodules, multiplication modules
2010 Mathematics Subject Classification:13C05,13C13
Received: 26 Nov. 2011 Revised 18 July 2012
* Correspondence Author Taheri@tmu.ac.ir

An R-module $M$ is said to be a *multiplication* module if for each submodule $N$ of $M$ there exists an ideal $I$ of $R$ such that $N = IM$. It is easy to see that in this case $N = (N :_R M)M$. Also the ideal $\theta(M)$ is defined as $\theta(M) := \sum_{m \in M}(Rm :_R M)$ . If $M$ is a multiplication module and $N$ is a submodule of it, then $M = \theta(M)M$ and $N = \theta\,(M)\,N$. (see [1]). An R-module $M$ is *sum-irreducible* if $M \neq 0$ and the sum of any two proper submodules of $M$ is always a proper submodule. Finally a proper submodule $N$ of an R-module $M$ is called a *prime submodule* if for each $r \in R$ the homothety $M/N \xrightarrow{r} M/N$ is either injective or zero. This implies that $Ann(M/N) = p$ is a prime ideal of $R$ , and $N$ is said to be a *p-prime submodule* (c.f. [7], [8], [10] and [11]).

## 2. Quasi-Secondary Submodules

The starting point of this section is the definition of quasi-secondary submodules of a module.

Definitian 2.1. Let $M$ be a non-zero R-module. Then the non-zero submodule $N$ of $M$ is said to be *quasi-secondary* if $\sqrt{(0 :_R N)} = p$ where $p$ is a prime ideal of R. It is obvious that every secondary (or second) submodule of a module is a quasi-secondary submodule, but the converse is not true in general. For example, 2Z is a 0-quasi-secondary submodule of the Z-module Z but it is not 0-secondary (or 0-second) submodule. (Here Z denotes the set of all integers.)

**Remark 2.2.**

(i) Let $M$ be a non-zero R-module and $N$ a submodule of it such that $\sqrt{(0 :_R N)} = m(m \in Max\,(R))$. Then $N$ is $m$-secondary ( $m$-second).

(ii) Every quasi-secondary submodule of a module over a zero-dimentional ring (i.e. , a ring in which every prime ideal is a maximal ideal) is secondary.

(iii) Every quasi-secondary submodule of a module over a D.V.R is secondary.

Definition 2.3. Let $M$ be an R-module and $N$ a submodule of $M$. An element $r$ of $R$ is called *co-primal* to $N$ if $rN = N$. Denote by $W(N)$ the set of all elements of $R$ that are not co-primal to $N$. The submodule $N$ is said to be a co-primal submodule of $M$ if $W(N)$ is an ideal of $R$. This ideal is always a prime ideal. In this case we say that $N$ is a *p-co-primal* submodule of M. The class of co-primal submodules of a module is a fairly large class. For example , all secondary (second) submodules are co-primal. Also it is easy to see that a sum-irreducible submodule of a module is co-primal. But, in general, a quasi-secondary submodule of a module may not be a co-primal submodule. (consider the Z-module Z.). It is worth to mention that in [2] the term secondal is used for co-primal submodules. The next proposition characterizes those p-quasi- secondary submodules which are p-co-primal.

Proposition 2 4. Let $N$ be a p-quasi-secondary submodule of an R-module M. Then $N$ is a pcoprimal submodule of $M$ if and only if it is a p-secondary submodule of $M$ .

Proof $\Rightarrow$) Let $N \xrightarrow{r} N$ be the R-endomorphism of $N$ given by multiphication by r of R and $rN \neq N$. Then by our assumption $r \in p = \{ s \in R \mid s N \neq N\}$. On the other hand, $p = \sqrt{0 :_R N}$ and so there exists a positive integer t such that $r^t N = 0$. The result follows. $\Leftarrow$) Is obvious.

The proof of two next propositions is easy and so we state them without proof.

Proposition 2.5. Let $M$ be a module over an integral domain and $N$ be a 0- co-primal submodule of $M$ .Then N is 0-secondary.

Proposition 2.6. Let $M$ be an R-module and $N_1, N_2, \dots, N_t$ be submodules of $M$. Then

(i) Suppose that for $i = 1, 2, N_i$ is $p_i$-quasi-secondary. Then $N_1 + N_2$ is quasi-secondary if and only if $p_1 \subseteq p_2$ or $p_2 \subseteq p_1$

(ii) If $N_1, \dots, N_t$ are p-quasi-secondary, then $N_1 + \cdots + N_t$ is a p-quasi-secondary submodule of $M$.

(iii) If $N_1 + \cdots + N_2$ is a p-quasi-secondary submodule of M.Then $N_j$ is p-quasi-secondary for some $j$ , $1 \leq j \leq t$.

## 3. Multiplication Modules

In this short section we give a property of quasi-secondary submodules of a multiphication module .

**Lemma 3.1.** let $M$ be a multiplication module and $N$ be a p-quasi-secondary submodule of $M$. Then $\theta(M) \nsubseteq p$.

**Proof.** Suppose that $\theta(M) \subseteq p$ and $0 \neq n \in N$. Then $Rn = \theta(M)Rn \subseteq pn$. Hence $n = p_0 n$ for some $p_0 \in p$. By our assumption there exists a positive integer t such that $p_0^t N = 0$. Therefore $n = p_0^t n = 0$, a contradiction.

**Theorem 3.2.** Suppose that $M$ is a faithfull multiplication module and $N$ a $p$-quasi-secondary submodule of $M$. Then $pM$ is a prime submodule of $M$.In particular, if $p \in \max(R)$, then $pM$ is a maximal submodule of $M$.

**Proof.** By Lemma 3.1 , $\theta(M) \not\subseteq p$. Now suppose that $pM = M = RM$. Then by $[1, \text{Theorem1.5}]$ $R \cap \theta(M) = \theta(M) = p \cap \theta(M)$and hence $\theta(M) \subseteq p$which is a contradiction. Thus $pM \neq M$ and the result of the first part follows from [6, Lemma 2.4(2)]. The last part can be deduced from the first part and [6, Corollary 2.7]

## Acknowledgement

The author would like to thank Kharazmi University for the financial support.

## References

1. Y. Al-shaniafi and S. Singh, "A companion ideal of a multiplication module", Periodica Mathematica Hungarica., 46 (1) (2003) 1-8.
2. H. Ansari-Toroghy and F. Farshadifar, "The dual notions of some generalizations of prime submodules", J. Algebra, 39 (2011) 2396-2416.
3. S. Ebrahimi Atani and A.Y.Darani, "On quasi-primary submodules", Chiang May J.Sci., 33(3) (2006) 249-254.
4. L. Fuchs, "On quasi-primary ideals", Acta.Sci.Math. (Szeged), 11 (1947) 174-183.
5. L. Fuchs and E. Mosteig, "Ideals theory in prufer domains", J. Algebra, 252 (2002) 411-430.
6. S. C. Lee, S. Kim, S. C. Chung, "Ideals and submodules of multiplication modules", J. Korean Math. Soc. 42 (5) (2005) 933-948.
7. C. P. Lu, "Prime submodules of modules", Comm. Math. Univ. Sancti. Pauli, 33 (1984) 61-69.
8. C. P. Lu, "Spectra of modules, Comm.in Algebra, 23 (1995) 3741-3752.
9. I. G. Macdonald, "Secondary representation of modules over commutative rings", Symp. Math. XI (1973) 23-43.
10. A. Marcelo, J. Masque, "Prime submodules, the descent invariant and modules of finite length", J.of Algebra, 189(1997) 273-293.
11. R. L. McCasland, P.F. Smith, "Prime submodules of Noetherian modules", Rocky Mtn. J. 23 (1993) 1041-1062.
12. S. Yassemi, The dual notion of prime submodules, Arch. Math. (Brno) 37 (2001) 273-278.