Hypergroup Structures with Regular Multiplications

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Abstract

I In Banach algebras, the group algebra L(G) is Arens regular if and only if G is finite. In this paper, the researcher has obtained a hypergroup structure (in the sense of Dunkl) whose measure algebra has regular multiplication. The most interesting result was that if L(X) is Arens regular then the convolution is Arens regular as a bilinear map. The condition obtained gives regularity of multiplication in the Hypergroup, which X is not finite.

Introduction

The regularity of a bounded bilinear mapping was defined by Arens (see [1]). For some important Banach algebras, the first and the second Arens product, on their second dual, are different. Therefore, these algebras are not Arens regular. A number of Banach algebras, commonly occuring in functional and harmonic analysis, are not Arens regular. The group algebra Ll(G)of a locally compact Hausdorff group is Arens regular if and only if G is finite (see [4], [13], [14]). In [9], the researcher has shown that the hypergroup algebra L(X), where X is a locally compact Hausdorff space, can be Arens regular without X being finite. Also, in [10], for a general measure algebra £, in M(X), if $e \in X$ is not isolated in supp £, and that δ_e acts as an identity for £, then £, is not Arens regular. These are not the only ways to construct the regular or irregular multiplications. In [8], for the circle group T, two multiplications have been constructed on M(T), one of which is regular and the other is irregular. In the present note, we obtain a hypergroup structure whose measure algebra has regular multiplication. Some related results can be

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found in [8],[9],[10]. Also, basic facts about measure algebras on hypergroups can be found in [6], [7], [11].

The researcher will begin with the Arens multiplications and the hypergroup structures.

1- Other Versions of Arens Multiplications

In [1], Arens showed how the multiplication of a Banach algebra could be extended to a multiplication on the second dual. His method was essentially algebraic, and this is indeed the easiest way to prove that the construction works. However, we shall describe the results. See [12] for details.

Let A be a set with a multiplication $(x, y) \to x.y.$ Let B be a set with $A \subseteq B$, under its topology, A is dense in B. For x, y in B, take $(\tau_\alpha),(s_\beta)$ in A such that $\lim_\alpha \tau_\alpha = x, \lim_\beta s_\beta = y.$ Then, the first extension of multiplication is given by

$$\tau_\alpha.y = \lim_\beta \tau_\alpha.s_\beta, \qquad x, y = \lim_\alpha \lim_\beta \tau_\alpha.s_\beta,$$

while the second extension is given by

$$x \odot s_\beta = \lim_\alpha \tau_\alpha.s_\beta, \qquad x \odot y = \lim_\alpha \tau_\alpha.s_\beta.$$

The set A is Arens regular (or, briefly, regular) if x, y=x ⊙y. By the above definition for a and b in B, the product a.b (resp. a ⊙ b) is continuous in the a (resp. b) variable for each fixed b (resp. a) in B. Generally, a.b (resp. a ⊙b) is not continuous in b (resp. a) when $a \notin A$ (resp. $b \notin A$). This suggests the firs result about regularity, which is entirely elementary.

Proposition 1.1. Let $A \subseteq B$ and let A be dense in B. Then.

(i) if A is commutative then A is Arens regular if and only if B is commutative.

(ii) A is Arens regular if and only if the multiplication in B is continuous in each variable (without any restriction on the other).

(iii) Let B be compact. Then, A is Arens regular if and only if the multiplication in B separately sequentially continuous.

Proof. See [5] and [12].

Regularity of Banach algebras 1.2. Let A be a normed space over K (K=R or K=C). The dual space A; i.e. A^*, is the vector space $\Lambda\ (A^*, K)$ equipped with the norm

$$\|f\| = \sup\{|f(x)| : x \in A, \|X\| \le 1\}.$$

Thus, A^* is a Banach space. Let $(A^{*})^*$ be the dual space of A^*, $(A^{*})^* = \Lambda(A^*, K)$. Since A^* is itself a Banach space, it is susceptible to the same construct; i.e. one can form $(A^{*})^* = A^{**}$; this is also a Banach space, called the dual or bidual of A, and denoted by A^{**}. This can go on.

For each $x \in A$, the value of an element $x^{**} \in A^{**}$ is defined by $x^{**}(f)=f(x)$ for all $f \in A^*$. So x^{**} is linear on A^*, and $\|x^{**}\| = \|x\|$. Thus, the canonical embedding mapping $x \to x^{**}$ preserves norms and an isometric from A into its second dual A^{**}. Therefore, we can regard A as a subspace of A^{**}.

Let $\sigma(A^{**}, A^*)$ be the weak*- topology on A^{**}. By [3], A is weak*-dense in A^{**}So, for F$\in A^{**}$, G$\in A^{**}$, we can find two bounded nets $(\mu_\alpha), (\nu_\beta)$ in A with F$= \omega^* - \lim_\alpha \mu_\alpha, G = \omega^* - \lim_\beta \nu_\beta$. The topological extension of first and second Arens product are given by

$$FG = \omega^* - l\underset{\alpha}{im}\,\omega^* - \lim_\beta \mu_\alpha \nu_\beta, \qquad F \circ G = \omega^* - l\underset{\beta}{im}\,\omega^* - \lim_\alpha \mu_\alpha \nu_\beta.$$

Thus, the order in which the limits are taken distinguishes between the extensions. Moreover, the first Arens product is characterized by the two properties:

(i) for each $G \in A^{**}$, the map F$\to$FG is weak*-continuous on A^{**}.

(ii) For each $\mu \in A$, the map G$\to \mu G$ is weak*- continuous on A^{**}.

The second Arens product is defined similarly. Therefore, the second dual $A^{**}$of A can be given the Banach algebra structure by means of the firs (or second) Arens product.

Now, we want to describe Arens products as an algebraic extension. Indeed, for F, G$\in A^{**}$, f$\in A^*$, and $\mu, \nu \in$ A, one can find FG, FoG successively as follows:

$$< FG, f > = < F, Gf >, < Gf, \mu > = < G, f\mu >, < f\mu, \nu > = < f, \mu\nu >,$$
$$< F \circ G, f > = < G, f \circ F >, < f \circ F, \mu > = < F, \mu \circ f >, < \mu \circ f, \nu > = < f, \nu\mu >.$$

So, a Banach algebra is said to have regular multiplication if FG=F$\circ$G.

A^{**} is not compact in the weak*-topology. But the closed until ball of A^{**} is weak-compact [3]. So, by definition or [5], we have:

Proposition 1.3. Let A be commutative. A is Arens regular if and only if the first or second Arens product in A^{**} is weak*-continuous in each variable.

Proof. For F, G$\in$ A**, There are two nets (μ_α) and (ν_β) in A which weak*-converge to F and G. So, A is Arens regular if and only if FG=F$\circ$G. It is equivalent to this fact; for all f$\in A^*$,

$$\lim_\alpha \lim_\beta f(\mu_\alpha \nu_\beta) = \lim_\beta \lim_\alpha f(\mu_\alpha \nu_\beta).$$

2- Hypergroup Structures

Let X be a locally compact Hausdorff space and M (X) denotes the set of all bounded, regular, complex Borel measures on X. For each μ and ν in M (X), $\mu * \nu$ denotes the convolution of μ and ν. Letδ, be the unit mass at r. The product formulas of the type.

$$\mu * \nu(f) = \int_x \int_x (\delta_X * \delta_y)(f) d\mu(x) d\nu(y)$$

On M (X) becomes a Banach algebra. Dunk1 (1972) and Jewett. (1975) have shown how one defines a product on M (X), which makes it a Banach algebra. In some cases, an investigation begins with a convolution algebra of measures as the primitive object, upon which to build a theory; this is the case of the analysis of the objects called hypergroups which are generalizations of the convolution algebra of Borel measures on a group. One of the objects of this paper will be the introduction of a large class of new convolution structures, many of which are new hypergroups.

Let C_b (X), C_0 (X) and C_c (X) denote the spaces of continuous functions on X which are bounded, those which vanish at infinity and those having compact support respectively. By M (X) and M_P(X), we abbreviate the space of Radon measures and probability measures on X.

Definition 2.1. A hypergroup $(X,*)$ is a Banach algebra of the Borel measures M (X) on a locally compact Hausdorff space X with product * called convolution it satisfies the following axioms:

(i)There is a map λ: $X \times X \to M_P(X)$ with for every $x, y \in X$, the measures $\lambda_{(x,y)}$ have compact supports and $\lambda_{(x,y)} = \lambda_{(y,x)}$;

(ii) for each f$\in C_C(X)$, the map (x, y)$\rightarrow \lambda_{(x,y)}(\int)$ is in $C_b(X\times X)$ and $r\rightarrow \lambda_{(x,y)}(\int)$ is in $C_c(X)$, for every $y\in x$;

(iii) the convolution $(\mu,\nu)\rightarrow \mu*\nu(or\,\mu\nu)$ of measures defined by

$$\mu*v(f)=\int_X\int_X \lambda_{(x,y)}(f)d\mu(x)dv(y), \qquad (\mu,v\in M(X),\int\in C_O(X))$$

is associative (and clearly $\lambda_{(x,y)}=\delta_X*\delta_y$)

(iv) there is a unique e$\in$ X such that $\lambda_{(x,y)}=\delta_x$ for all x$\in$ X.

In [7], Ghahramani and Medghalchi have constructed and studied the subalgebra of M (X) which is determined in the following way:

$$L(X)=\{\mu\in M(X): x\rightarrow|\mu|*\delta_{x,}x\rightarrow\delta_x*|\mu| \text{ are norm-continuous}\}.$$

This algebra generalizes the algebra L (G) of M (G) for the locally compact topological groups G.They have shown that L (X) is a Banach subalgebra of M (X) and it has a bounded approximate identity of norm 1. Therefore, L (X) can be regarded as a subspace of L $(X)^{**}$ and then L (X) is weak*-detise in L $(X)^{**}$ in [11], Medghacli studied the second dual of L (X).

Let A, B, C, be disjoint. And e, z be single points not in $A\bigcup B\bigcup C$.Write X=$\{e\}\bigcup A\bigcup B\bigcup C\bigcup\{Z\}$. Let X be a compact Hansdorff space with e and z as isolated pointes. Each $\mu\in$ M (X) can be written in the unique form

$$\mu_e\delta_e+\mu_B+\mu_e+\mu_s\delta_s$$

where, μ_A,μ_B,μ_C are the restriction of μ to A, B, C, respectively, and μ_e and μ_z are scalars.

The following Theorem givens the structure of a hypergroup on the locally com. Pact space X.

Theorem 2.2. Let $\lambda: A\times B\rightarrow$(M$_y$(C), weak*) be continuous and the map $x\rightarrow\lambda_{(x,y)}$(f) is in C$_c$(C) for every x, y $\in A\bigcup B, f\in$ C(C) and $\lambda(a,b)=\lambda(b,a)$ $(a\in A,b\in B)$. There is a hypergroup structure on X such that M (X-{e}) =L (X) if and only if $\mathrm{M}(A\bigcup B)\subseteq L(X)$.

Proof. We define a multiplication on M (X) by the following way:

take a map $\tilde{\lambda}: X\times X\rightarrow M_P(X)$ with two properties.

(i) $\tilde{\lambda}_{(x,e)}=\tilde{\lambda}_{(e,x)}=\delta_x(x\in X)$;

(ii) $\tilde{\lambda}_{(x,y)}=\begin{cases}\lambda_{(x,y)} & (x\in A, y\in B)\\ \delta_Z & otherwise\end{cases}$

it is clear that $\left\|\tilde{\lambda}_{(x,y)}\right\|=1$ for all x, y∈ X.

Then, we define a multiplication $\mu\nu, for\, \mu,\nu\in M(X), by$

$\mu\nu=\int_X\int_X \tilde{\lambda}_{(x,y)}\, d\mu(x)dv(y)$

It is easy to show that the multiplications is commutative and is the identity. Now, we prove that it is associative.

If one of the x, y, z, (inX) is equal to e, then

$$(\delta_x*\delta_y)*\delta_z=\delta_x*(\delta_y*\delta_x)$$

Now, suppose that μ,ν,ξ are in M (X-{e}). Therefore,

$$\mu=\mu_A+\mu_B+\mu_C+\mu_Z\delta_Z,$$
$$\nu=\nu_A+\nu_B+\nu_C+\nu_z\delta_z,$$
$$\xi=\xi_A+\xi_B+\xi_C+\xi_z\delta_{z.}$$

Suppose Y= $A\cup C\cup\{z\}$. We have

$$\begin{aligned}\mu_A\nu_Y&=\int_X\int_X \tilde{\lambda}(x,y)d\mu_A(x)dvY(y)\\&=\int_X\int_X \delta_z d\mu_A(x)dvY(y)\\&=\mu_A(I)\nu_Y(I)\delta_Z.\end{aligned}$$

Similarly $\mu_Y\nu_A=\mu_Y(1)\nu_A(1)\delta_Z$ $and\ if\ T=B\cup C\cup\{z\}then$

$\mu_T\nu_B=\mu_T(1)\nu_B(1)\delta_Z,\mu_B\nu_T=\mu_B(1)\nu_T(1)\delta_Z.$

Let‘’*’’be the convolution arising from the multiplication in A and B(or in B and A) i.e.,

$$\mu_A*\nu_B=\mu_A\nu_B=\int_X\int_X \tilde{\lambda}(x,y)d\mu_A(y)dv_B(x),$$
$$\mu_A*\nu_B(f)=\int_A\int_B\int_C f(t)d\lambda_{(x,y)}(t)dv_B(x)d\mu_A(y).$$

Hence, supp $(\mu_A*\nu_B)\subseteq C.$ Therefore, we have

$$\begin{aligned}\mu\nu&=(\mu_A+\mu_B+\mu_C+\mu_z\delta_z)(\nu_A+\nu_B+\nu_C+\nu_z\delta_z)\\&=\mu_A\nu_Y+\mu_A*\nu_B+\mu_B\nu_T+\mu_B*\nu_A+\mu_z\delta_z\nu\\&=\mu_A(1)\nu_Y(1)\delta_z+\mu_B(1)\nu_T(1)\delta_z+\mu_z\delta_z(1)\nu(1)\delta_z+\mu_A*\nu_B+\mu_B*\nu_A\\&=[\mu(1)\nu(1)-\mu_A(1)\nu_B(1)-\mu_B(1)\nu_A(1)]\delta_z+\mu_A*\nu_B+\mu_B*\nu_A\end{aligned}$$

On the other hand,

$$\mu_A * \nu_B(1) = \int_A \int_B \int_C \; d\lambda_{(x,y)}(t) d\nu_B(x) d\mu_A(y).$$
$$= \int_A \int_B d\nu_B(x) d\mu_A(y) = \mu_A(1)\nu_B(1).$$

Hence,

$$(\mu\nu)\ \psi = [\mu(1)\nu(1) - \mu_A(1)\nu_B(1) - \mu_B(1)\nu_A(1)]\delta_z\ \psi + (\mu_A * \nu_A + \mu_B * \nu_A)\ \psi$$
$$= \mu(1)\nu(1)\psi\ (1)\delta_z.$$

Similarly $\mu(\nu\psi) = \mu(1)\nu(1)\ \psi(1)\delta_z$ So that, the multiplication is associative

Now, let $f \in C_c(X)$, then we have

$$\int_X f(t) d\tilde{\lambda}_{(x,e)}(t) = f(x),$$

$$\int_X f(t) d\tilde{\lambda}_{(x,y)}(t) = \begin{cases} d\lambda_{(x,y)}(f) & (x \in A, y \in B) \\ f(z) & otherwise. \end{cases}$$

Then the map (x, y) $\rightarrow \tilde{\lambda}_{(x,e)}$ (f) is in C_b (X×X) and the map $x \rightarrow \tilde{\lambda}_{(x,y)}$ (f) is in C_c (X) for all $y \in X$. Therefore, X is a hypergroup.

We now prove that L (X)=M (X-{e}). Since X is not discrete, $\delta_e \notin$ L (X) (Theorem 2, [9]). Let $\mu \in$ M (X-{e}). Then

$$\delta_x|\mu| = \begin{cases} |\mu| & (x = c) \\ |\mu|(1)\delta_z & (x = z). \end{cases}$$

Now, if $x \in X - \{e, z\}$ then,

$$\delta_x|\mu| = \begin{cases} |\mu|(1) - |\mu_B|(1)\,\delta_z + \delta_x * |\mu_B| & (x = A) \\ |\mu|(1) - |\mu_A|(1)\,\delta_z + \delta_x * |\mu_A| & (x = B). \end{cases}$$

Hence, for x=y=z or x=y=e,

$$\|\delta_x|\mu| - \delta_y|\mu|\| = 0$$

Otherwise,

$$\|\delta_x|\mu| - \delta_y|\mu|\| = \begin{cases} \|\delta_x|\mu_B| - \delta_y|\mu_B|\| & (x, y \in A) \\ \|\delta_x|\mu_A| - \delta_y|\mu|_A\| & (x, y \in B). \end{cases}$$

Therefore, $\mu \in$ L (X) if and only if $\mu_A \in L(X), \mu_B \in L(X)$. This statement is equivalent to, $\mu \in$ L (X) if and only if M (A$\cup$B)$\subseteq$L (X). So, the conclusion holds.

Let M (X) be the space of bounded regular Borel measures. We shall say that M (X) has a general measure multiplication, if there exists a bilinear associative map

$\phi : M(X)\times M(X)\to M(x)$ such that

$$\phi(M_P(X)\times M_P))\subseteq M_P(X).$$

Also, we shall say that ϕ is Arens regular, if for every two nets $(\mu_a),(v_\beta)$ in M(X),

$$w^*-\lim_\alpha w^*-\lim_\beta \phi(\mu_\alpha,v_\beta)=w^*-\lim_\beta w^*-\lim_\beta \phi(\mu_\alpha,v_\beta),$$

when, both exist (see introduction [10]).

Theorem2.3. Let $M(A\bigcup B)\subset L(X)$. Then, L(X) is Arens regular if and only if there is a bilinear map ϕ : M(A)×M(B)→M(C) which is Arens regular.

Proof. By theorem1, L(X)=M(X-{e}) and (by the lemma, 4, [9]),

$$L(X)=M(A)\oplus M(B)\oplus M(C)\oplus \square\,\delta_z,$$
$$L(X)^{**}=M(A)^{**}\oplus M(B)^{**}\oplus M(C)^{**}\oplus \square\,\delta_z.$$

So, each $\mu\in$ L(X), F∈ $L(X)^{**}$, can be written uniquely in the form

$$\mu=\mu l_A+\mu_B+\mu_C+\mu_z\delta_{z,}$$
$$F=F_A+F_B+F_C+\mu_z\delta_{z,}$$

where, μ_A is the restriction of μ to A and F_A is the restriction of F to $M(A)^*$ and so on.

Let $\mu,v\in$ M(X). We define Φ : M(X)×M(X)→M(X) by

$$\Phi(\mu,v)=\mu v=\int_X\int_X \tilde{\lambda}(x,y)d\mu(x)dv*y).$$

If $\mu,v\in M_P$(X) then $\phi(\mu,v)\in M_P$(X). Thus, the multiplication Φ maps probability measures to probability measures.

First, suppose that L(X) is Arens regular. So, for every nets $(\mu_\alpha)\subset$ M(A), $(v\beta)\subset M(B), if$

$$w^*-\lim_\alpha w^*-\lim_\beta \Phi(\mu_\alpha,v_\beta),w^*-\lim_\beta w^*-\lim_\alpha \Phi(\mu_\alpha,v_\beta)$$

exist, and then they are equal (by Theorem 1, [5]).

Conversely, let Φ be Arens regular and F,G∈ $L(X)^{**}$. There are two nets $(\mu_\alpha) and\, (v_\beta)$ in L(X) whit

$$w^*-\lim_\alpha \mu_\alpha=F,\ w^*-\lim_\beta \mu_\beta=G,$$
$$\operatorname{supp}\mu_\alpha\subseteq X-\{e\},\ \operatorname{supp}\mu_\beta\subseteq X-\{e\}$$

The multiplication Φ is Arens regular. Therefore,

$$w^*-\lim_\alpha w^*-\lim_\beta \phi((\mu_A)_\alpha(v_B)_\beta)=w^*-\lim_\beta w^*-\lim_\alpha \phi((\mu_A)_\alpha(v_B)_\beta),$$
$$w^*-\lim_\alpha w^*-\lim_\beta \phi((v_B)_\alpha(\mu_A)_\alpha)=w^*-\lim_\beta w^*-\lim_\alpha \phi((v_B)_\alpha(\mu_A)_\alpha).$$

Combining the above equalities, we have

$$FG = w^* - \lim_{\alpha} w^* - \lim_{\beta} \mu_\alpha \nu_\beta$$
$$= w^* - \lim_{\alpha} w^* - \lim_{\beta} [(\mu_\alpha(1)\nu_\beta(1) - (\mu_A)_\alpha(1)(\nu_B)_\beta(1) - (\mu_B)_\alpha(1)(\nu_A)_\alpha(1))\delta_z +$$
$$\phi((\mu_A)_\alpha(\nu_B)_\beta) + \phi((\mu_B)_\alpha(\nu_A)_\beta)] = w^* - \lim_{\beta} w^* - \lim_{\alpha} \mu_\alpha \nu_\beta = GF.$$

Thus, FG=GF. By (Propositon1, [5]), L(X) is Arens regular.

Now, let A and C be disjoint, X=$\{e\} \cup A \cup C \cup \{z\}$ and $e, z \notin A \cup C$. Whit the topology of X, A and C are compact subspaces of X and e, z are isolated points.

Theorem2.4. If $\lambda : A \times A \to M_P(X)$ is weak*-continuous and symmetric, then there exists a hypergroup structure on X so that

(i) M(A)$\subseteq$L(X) if and only if M(X-{e})=L(X);

(ii) Let M(A) $\subseteq$L(X). Then there exists a bilinear associative map

Φ : M(A) $\subseteq$ M(A)$\to$M(C) which maps probability measures to probability measures and L(X) is Arens regular if and only if Φ is Arens regular.

Proof. Define $\tilde{\lambda} : X \times X \to M_P(X)$ with the following equations:

(i) $\tilde{\lambda}_{(x,e)} = \tilde{\lambda}_{(e,x)} = \delta_x$;

(ii) $\tilde{\lambda}_{(x,y)} = \begin{cases} \lambda^{(x,y)} & (x, y \in A) \\ \delta_z & (otherwise). \end{cases}$

Then, we define a convolution $\mu\nu$ *for* $\mu, \nu \in M(X)$ *by*

$$\mu\nu = \int_X \int_X \tilde{\lambda}_{(x,y)}\, d\mu(x) d\nu(y).$$

It is clear that, if $\mu, \nu \in$ M(X-{e}) and then

$$\mu\nu = (\mu(1)\nu(1) - \mu_A(1)\nu_A(1)\delta_z + \mu_A * \nu_A.$$

The rest of the proof is the same as the proof of last theorem.

Let G be an arbitrary locally compact Housdorff group and μ be a right invariant Haar measure on G. The space L^1(G) of integrable functions on G, with the convolution taken as product, is a Banach Algebra. P. Civin and B. Yood [4] have shown that; if G is commutative and infinite set then the Banach Algebra L^1(G) is not Arens regular. N.Young [14] has extended this result to non-commutative case. A.Ulger [13] has presented a very simple proof of the Theorem, which says that the group algebra $L^1(G)$ is Arens regular if and only if G is finite. In this paper we present a Theorem, which shows that the Young's result does not hold in a hypergroup. This theorem is an application of Theorem2.2.

Theorem2.4. There is a hypergroup algebra M (X) which has regular multiplication and L(X) is Arens regular but X is not finite.

Construction. Let $A=[a_1,b_1]$, $B=[a_2,b_2]$, C={a,b} be subsets of an ordered set in the ordered topology and A,B,C are disjoint. Let $X=\{e\}\cup A\cup B\cup C\cup\{z\}$, with the topology in which e, a, b, z are isolated points and supposed $\phi: X\to[0,1]$ is a continuous function with supp ϕ=X. Define $\lambda: A\times B\to M_P(C)$ by

$$\lambda_{(x,y)}=\phi(x)\phi(y)\delta_a+(1-\phi(x)\phi(y))\delta_b.$$

So, $\lambda_{(x,y)}=\lambda_{(y,x).}$ *If* $f\in C_O(X)(-C_c(X))$ *then,*

$$\lambda_{(x,y)}(f)=\int_X f(t)d\lambda_{(x,y)}(t)=\phi(x)\phi(y)f(a)+(1-\phi(x)\phi(y))f(b)$$

Hence, if $\{(x_n, y_n)\}$ is sequence converge to (x, y) then

$$\lim_n \lambda_{(x_n,y_n)}(f)=\lambda_{(x,y)}(f).$$

Therefore, $\lambda: A\times B\to(M_P(C),Weak^*)$ is continuous. By Theorem 2.2 X is a hypergroup.

To prove M(A$\cup$B)$\subseteq$L(X), let $\mu\in$ M(A$\cup$B). Then, $\mu=\mu_A+\mu_B$. Suppose that x$\in$ B, then,

$$\begin{aligned}\delta_x|\mu_A| &= \int_X\int_X \lambda_{(u,v)}d\delta_x(v)d|\mu_A|(u)\\ &= \int_X \lambda_{(u,x)}d|\mu_A|(u),\end{aligned}$$

So, for x, y$\in$ B,

$$\begin{aligned}\|\delta_X|\mu_A|-\delta_y|\mu_A|\| &= \left\|\int_X \lambda_{(u,x)}-\lambda_{(u,y)}d|\mu_A|(u)\right\|\\ &= |\phi(x)-\phi(y)|\left\|\int_X \phi(u)d|\mu_A|(u))\delta_a-\int_X\phi(u)d|\mu_A|(u))\delta_b\right\|.\\ &\le 2|\phi(x)-\phi(y)|\|\mu_A|(\phi),\end{aligned}$$

ϕ is continuous, so $\mu_A\in$ L(X). Similarly, $\mu_B\in$ L(X). Then L(X)=M(X-{e}).

We now prove that ϕ: M(A)×M(B)→M(C) is Arens regular. First suppose that $\mu_A \in$ M(A), $\nu_B \in$ M(B). So,

$$\phi(\mu_A, v_B) = \mu_A * v_B = \int_X \int_X \lambda_{(x,y)} d\mu_A(x) dv_B(y)$$
$$= \int_X \int_X [\phi(x)\phi(y)\delta_a + (1-\phi(x)\phi(y)\delta_b] d\mu_A(x) dv_B(y)$$
$$= \mu_A(\phi) v_B(\phi)\delta_a + (\mu_A(1) v_B(1) - \mu_A(\phi) v_B(\phi))\delta_b.$$

Now, suppose $\{(\mu_A)_n\}, \{(\nu_B)_m\}$ are two sequences such that
w*-$\lim_n w^* - \lim_m \phi((\mu_A)_{n,} (v_B)_m), w^* - \lim_m w^* \lim_n \phi((\mu_A)_{n,} (v_B)_m)$ exist, then

$$w^* - l\,im_n\, w^* - \lim_m \phi((\mu_A)_{n,} (v_B)_m) = w^* - \lim_n w^* - \lim_m [(\mu_A)_n(\phi)(v_B)_m(\phi)\delta_a$$
$$+ ((\mu_A)_n(I)(\phi_B)_m(1) - (\mu_A)_n(\phi)(v_B)_m(\phi)\delta_b]$$
$$= w^* - \lim_m w^* - \lim_n \phi((\mu_A)_{n,} (v_B)_m).$$

Hence, ϕ is Arens regular. By Theorem 2.3, L(X) is Arens regular, but X is not finite.

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