

## Numerical Solution of Fuzzy Differential Equation by Runge-Kutta Method

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### Abstract

In this paper, the numerical algorithms for solving ‘fuzzy ordinary differential equations’ are considered. A scheme based on the 4<sup>th</sup> order Runge-Kutta method is discussed in detail and it is followed by a complete error analysis. The algorithm is illustrated by solving some linear and nonlinear fuzzy Cauchy problems.

### 1. Introduction

The concept of fuzzy derivative was first introduced by S.L. Chang, L.A. Zadeh in [1]. It was followed up by D. Dubois, H. Prade in [2], who defined and used the extension principle. The fuzzy differential equation and the initial value problem were regularly treated by O. Kaleva in [5] and [6], by S. Seikkala in [7]. The numerical method for solving fuzzy differential equations is introduced by M. Ma, M. Friedman, A. Kandel in [10] by the standard Euler method. The structure of this paper organizes as follows:

In section 2 some basic results on fuzzy numbers and definition of a fuzzy derivative, which have been discussed by S. Seikkala in [10], are given. In section 3 we define the problem, this is a fuzzy Cauchy problem whose numerical solution is the main interest of this work. The numerically solving fuzzy differential equation by 4<sup>th</sup> order Runge-Kutta method is discussed in section 4. The proposed algorithm is illustrated by solving some examples in section 5 and conclusion is in section 6.

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**Keywords:** Fuzzy Differential Equation, 4<sup>th</sup> Order Runge-Kutta Method, Fuzzy Cauchy Problem.



## 2. Preliminaries

Consider the initial value problem

$$\begin{cases} y'(t) = f(t, y(t)); & a \leq t \leq b, \\ y(a) = \alpha. \end{cases} \quad (1)$$

The basis of all Rung-Kutta methods is to express the difference between the value of  $y$  at  $t_{n+1}$  and  $t_n$  as

$$y_{n+1} - y_n = h \sum_{i=1}^m w_i k_i. \quad (2)$$

where the  $w_i$ 's for  $i = 1, 2, \dots, m$ , are constants and

$$k_i = f(t_n + \alpha_i h, y_n + h \sum_{j=1}^{i-1} \beta_{ij} k_j). \quad (3)$$

Equation (2) is to be coincident whit Taylor series order  $m$ . Therefore, the truncation error  $T_m$ , can be written as

$$T_m = \gamma_m h^{m+1} + O(h^{m+2}).$$

The true magnitude of  $\gamma_m$  will generally be much less than the bound of Theorem 2.1. Thus, if the  $O(h^{m+2})$  term is small compared with  $\gamma_m h^{m+1}$ , as we expect, to be so if  $h$  is small enough, then the bound on  $\gamma_m h^{m+1}$ , will usually be a bound on the error as a whole. The famous nonzero constants  $\alpha_i, \beta_{ij}$  in 4<sup>th</sup> order Runge-Kutta method are

$$\alpha_1 = 0, \alpha_2 = \alpha_3 = \frac{1}{2}, \alpha_4 = 1, \beta_{21} = \frac{1}{2}, \beta_{32} = \frac{1}{2}, \beta_{43} = 1,$$

and we have, see [9]

$$\begin{aligned} y_0 &= \alpha, \\ k_1 &= f(t_i, y_i), \\ k_2 &= f(t_i + \frac{h}{2}, y_i + \frac{h}{2} k_1), \\ k_3 &= f(t_i + \frac{h}{2}, y_i + \frac{h}{2} k_2), \\ k_4 &= f(t_i + \frac{h}{2}, y_i + \frac{h}{2} k_3), \\ y_{i+1} &= y_i + \frac{h}{6} (k_1 + 2k_2 + 2k_3 + k_4), \end{aligned} \quad (4)$$

where

$$a = t_0 < t_1 < \dots < t_N = b \text{ and } h = \frac{b-a}{N} = t_{i+1} - t_i. \quad (5)$$

**Theorem 2.1.** Let  $f(t, y)$  belong to  $C^4[a, b]$  and let its partial derivatives be bounded and assume there exist,  $P, M$ , positive numbers, such that

$$|f(t, y)| < M, \quad \left| \frac{\partial^{i+j} f}{\partial t^i \partial y^j} \right| < \frac{P^{i+j}}{M^{j-1}}, \quad i + j \leq m,$$

then in the 4th order Rung-Kutta method (Proof [9]).

$$y(t_{i+1}) - y_{i+1} = \frac{73}{720} h^5 MP^4 + O(h^6).$$

A triangular fuzzy number  $\nu$ , is defined by three numbers  $a_1 < a_2 < a_3$  where the graph of  $\nu(x)$ , the membership function of the fuzzy number  $\nu$ , is a triangle with base on the interval  $[a_1, a_3]$ , and vertex at  $x = a_2$ . We specify  $\nu$  as  $(a_1 / a_2 / a_3)$ . We will write: (1)  $\nu > 0$  if  $a_1 > 0$ ; (2)  $\nu \geq 0$  if  $a_1 \geq 0$ ; (3)  $\nu < 0$  if  $a_3 < 0$ ; and (4)  $\nu \leq 0$  if  $a_3 \leq 0$

Let  $E$  be the set of all upper semicontinuous normal convex fuzzy numbers with bounded r-level sets It means that if  $\nu \in E$  then the r-level set

$$[\nu]_r = \{s \mid \nu(s) \geq r\}, \quad 0 < r \leq 1,$$

is a closed bounded interval which is denoted by

$$[\nu]_r = [v_1(r), v_2(r)].$$

Let  $I$  be a real interval. A mapping  $x: I \rightarrow E$  is called a fuzzy process and its r-level set is denoted by

$$[x(t)]_r = [x_1(t; r), x_2(t; r)], \quad t \in I, \quad r \in (0, 1].$$

The derivative  $x'(t)$  of a fuzzy process  $x$  is defined by

$$[x'(t)]_r = [x'_1(t; r), x'_2(t; r)], \quad t \in I, \quad r \in (0, 1],$$

provided that this equation defines a fuzzy number, as in Sikkala [7].

**Lemma 2.1.** Let  $\nu, w \in E$  and  $s$  be a scalar (see [7]). Then for  $r \in (0, 1]$

$$[\nu + w]_r = [v_1(r) + w_1(r), v_2(r) + w_2(r)],$$

$$[\nu - w]_r = [v_1(r) - w_2(r), v_2(r) - w_1(r)],$$

$$[\nu \cdot w]_r = [\min\{v_1(r) \cdot w_1(r), v_1(r) \cdot w_2(r), v_2(r) \cdot w_1(r), v_2(r) \cdot w_2(r)\},$$

$$\max\{v_1(r) \cdot w_1(r), v_1(r) \cdot w_2(r), v_2(r) \cdot w_1(r), v_2(r) \cdot w_2(r)\}],$$

$$[s \cdot \nu]_r = s[\nu]_r.$$

### 3. A Fuzzy Cauchy Problem

Consider the fuzzy initial value problem

$$\begin{cases} y'(t) = f(t, y(t)), & t \in I = [0, T], \\ y(0) = y_0, \end{cases} \quad (6)$$

where  $f$  is a continuous mapping from  $R_+ \times R$  into  $R$  and  $y_0 \in E$  with  $r$ -level sets

$$[y_0]_r = [y_1(0; r), y_2(0; r)], \quad r \in (0, 1).$$

The extension principle of Zadeh leads to the following definition of  $f(t, y)$  when  $y = y(t)$  is a fuzzy number

$$f(t, y)(s) = \sup \{y(\tau) \mid s = f(t, \tau)\}, \quad s \in R.$$

It follows that

$$[f(t, y)]_r = [f_1(t, y; r), f_2(t, y; r)], \quad r \in (0, 1),$$

where

$$\begin{aligned} f_1(t, y; r) &= \min \{f(t, u) \mid u \in [y_1(t; r), y_2(t; r)]\}, \\ f_2(t, y; r) &= \max \{f(t, u) \mid u \in [y_1(t; r), y_2(t; r)]\}. \end{aligned} \quad (7)$$

**Theorem 3.1.** *Let  $f$  satisfy*

$$|f(t, v) - f(t, \bar{v})| \leq g(t, |v - \bar{v}|), \quad t \geq 0, \quad v, \bar{v} \in R,$$

where  $g : R_+ \times R_+ \rightarrow R_+$  is a continuous mapping such that  $r \rightarrow g(t, r)$  is nondecreasing, the initial value problem

$$u'(t) = g(t, u(t)), \quad u(0) = u_0, \quad (8)$$

has a solution on  $R_+$  for  $u_0 > 0$  and that  $u(t) = 0$  is the only solution of (8) for  $u_0 = 0$ .

Then the fuzzy initial value problem (6) has a unique fuzzy solution.

Proof [7].

### 4. 4<sup>th</sup> Order Runge-Kutta Method

Let the exact solution  $[Y(t)]_r = [Y_1(t; r), Y_2(t; r)]$  be approximated by some  $[y(t)]_r = [y_1(t; r), y_2(t; r)]$ . From (2),(3) we define

$$\begin{aligned}
 y_1(t_{n+1}; r) - y_1(t_n; r) &= h \sum_{i=1}^4 w_i k_{i,1}(t_n, y(t_n; r)), \\
 y_2(t_{n+1}; r) - y_2(t_n; r) &= h \sum_{i=1}^4 w_i k_{i,2}(t_n, y(t_n; r)),
 \end{aligned}
 \tag{9}$$

where the  $w_i$ 's are constants and

$$\begin{aligned}
 [k_i(t, y(t; r))]_r &= [k_{i,1}(t, y(t; r)), k_{i,2}(t, y(t; r))], \quad i = 1, 2, 3, 4 \\
 k_{i,1}(t_n, y(t_n; r)) &= f(t_n + \alpha_i h, y_1(t_n) + h \sum_{j=1}^{i-1} \beta_{i,j} k_{j,1}(t_n, y(t_n; r))),
 \end{aligned}
 \tag{10}$$

and

$$\begin{aligned}
 k_{i,2}(t_n, y(t_n; r)) &= f(t_n + \alpha_i h, y_2(t_n) + h \sum_{j=1}^{i-1} \beta_{i,j} k_{j,2}(t_n, y(t_n; r))) \\
 k_{1,1}(t, y(t; r)) &= \min\{f(t, u) \mid u \in [y_1(t; r), y_2(t; r)]\}, \\
 k_{1,2}(t, y(t; r)) &= \max\{f(t, u) \mid u \in [y_1(t; r), y_2(t; r)]\}, \\
 k_{2,1}(t, y(t; r)) &= \min\{f(t + \frac{h}{2}, u) \mid u \in [z_{1,1}(t; y(t; r)), z_{1,2}(t; y(t; r))]\}, \\
 k_{2,2}(t, y(t; r)) &= \max\{f(t + \frac{h}{2}, u) \mid u \in [z_{1,1}(t; y(t; r)), z_{1,2}(t; y(t; r))]\}, \\
 k_{3,1}(t, y(t; r)) &= \min\{f(t + \frac{h}{2}, u) \mid u \in [z_{2,1}(t; y(t; r)), z_{2,2}(t; y(t; r))]\}, \\
 k_{3,2}(t, y(t; r)) &= \max\{f(t + \frac{h}{2}, u) \mid u \in [z_{2,1}(t; y(t; r)), z_{2,2}(t; y(t; r))]\}, \\
 k_{4,1}(t, y(t; r)) &= \min\{f(t + h, u) \mid u \in [z_{3,1}(t; y(t; r)), z_{3,2}(t; y(t; r))]\}, \\
 k_{4,2}(t, y(t; r)) &= \max\{f(t + h, u) \mid u \in [z_{3,1}(t; y(t; r)), z_{3,2}(t; y(t; r))]\}.
 \end{aligned}
 \tag{11}$$

Where in the 4<sup>th</sup> order Runge-Kutta method,

$$\begin{aligned}
 z_{1,1}(t, y(t; r)) &= y_1(t; r) + \frac{h}{2} k_{1,1}(t, y(t; r)) \quad , \quad z_{1,2}(t, y(t; r)) = y_2(t; r) + \frac{h}{2} k_{1,2}(t, y(t; r)), \\
 z_{2,1}(t, y(t; r)) &= y_1(t; r) + \frac{h}{2} k_{2,1}(t, y(t; r)) \quad , \quad z_{2,2}(t, y(t; r)) = y_2(t; r) + \frac{h}{2} k_{2,2}(t, y(t; r)), \\
 z_{3,1}(t, y(t; r)) &= y_1(t; r) + h k_{3,1}(t, y(t; r)) \quad , \quad z_{3,2}(t, y(t; r)) = y_1(t; r) + h k_{3,2}(t, y(t; r)).
 \end{aligned}
 \tag{12}$$

Define,

$$\begin{aligned}
 F[t, y(t; r)] &= k_{1,1}(t, y(t; r)) + 2k_{2,1}(t, y(t; r)) + 2k_{3,1}(t, y(t; r)) + k_{4,1}(t, y(t; r)), \\
 G[t, y(t; r)] &= k_{1,2}(t, y(t; r)) + 2k_{2,2}(t, y(t; r)) + 2k_{3,2}(t, y(t; r)) + k_{4,2}(t, y(t; r)).
 \end{aligned}
 \tag{13}$$

The exact and approximate solutions at  $t_n, 0 \leq n \leq N$  are denoted by

$[Y(t_n)]_r = [Y_1(t_n; r), Y_2(t_n; r)]$  and  $[y(t_n)]_r = [y_1(t_n; r), y_2(t_n; r)]$ , respectively. The solution is calculated at grid points of (5). By (9),(13) we have

$$\begin{aligned}
 Y_1(t_{n+1}; r) &\approx Y_1(t_n; r) + \frac{h}{6} F[t_n, Y(t_n); r], \\
 Y_2(t_{n+1}; r) &\approx Y_2(t_n; r) + \frac{h}{6} G[t_n, Y(t_n); r].
 \end{aligned}
 \tag{14}$$

We define

$$\begin{aligned} y_1(t_{n+1};r) &= y_1(t_n;r) + \frac{h}{6} F[t_n, y(t_n);r], \\ y_2(t_{n+1};r) &= y_2(t_n;r) + \frac{h}{6} G[t_n, y(t_n);r], \end{aligned} \tag{15}$$

the following lemmas will be applied to show convergence of these approximates,i.e.,

$$\begin{aligned} \lim_{h \rightarrow 0} y_1(t;r) &= Y_1(t;r), \\ \lim_{h \rightarrow 0} y_2(t;r) &= Y_2(t;r). \end{aligned}$$

**Lemma 4.1.** *Let the sequence of numbers  $\{W_n\}_{n=0}^N$  satisfy*

$$|W_{n+1}| \leq A|W_n| + B, \quad 0 \leq n \leq N-1,$$

for some given positive constants  $A$  and  $B$  (Proof [10]). Then

$$|W_n| \leq A^n |W_0| + B \frac{A^n - 1}{A - 1}, \quad 0 \leq n \leq N.$$

**Lemma 4.2.** *Let the sequence of numbers  $\{W_n\}_{n=0}^N$ ,  $\{V_n\}_{n=0}^N$  satisfy*

$$|W_{n+1}| \leq |W_n| + A \cdot \max\{|W_n|, |V_n|\} + B,$$

$$|V_{n+1}| \leq |V_n| + A \cdot \max\{|W_n|, |V_n|\} + B,$$

for some given positive constants  $A$  and  $B$ , and denote

$$|U_n| = |W_n| + |V_n|, \quad 0 \leq n \leq N.$$

Then

$$|U_n| \leq \bar{A} |U_0| + \bar{B} \frac{\bar{A}^n - 1}{\bar{A} - 1}, \quad 0 \leq n \leq N,$$

where  $\bar{A} = 1 + 2A$  and  $\bar{B} = 2B$  (Proof [10]).

Let  $F(t, u, v)$  and  $G(t, u, v)$  be obtained by substituting  $[y(t)]_r = [u, v]$  in (13).

$$F[t, u, v] = k_{1,1}(t, u, v) + 2k_{2,1}(t, u, v) + 2k_{3,1}(t, u, v) + k_{4,1}(t, u, v),$$

$$G[t, u, v] = k_{1,2}(t, u, v) + 2k_{2,2}(t, u, v) + 2k_{3,2}(t, u, v) + k_{4,2}(t, u, v).$$

The domain where  $F$  and  $G$  are defined is therefore

$$K = \{(t, u, v) \mid 0 \leq t \leq T, -\infty < v < \infty, -\infty < u \leq v\}.$$

**Theorem 4.1.** Let  $F(t, u, v)$  and  $G(t, u, v)$  belonged to  $C^4(K)$  and let the partial derivatives of  $F$  and  $G$  be bounded over  $K$ . Then, for arbitrary fixed  $r : 0 \leq r \leq 1$ , the approximative solutions (14) converge to the exact solutions  $Y_1(t; r)$  and  $Y_2(t; r)$  uniformly in  $t$ .

**Proof.** It is sufficient to show

$$\lim_{h \rightarrow 0} y_1(t_N; r) = Y_1(t_N; r),$$

$$\lim_{h \rightarrow 0} y_2(t_N; r) = Y_2(t_N; r),$$

where  $t_N = T$ . For  $n = 0, 1, \dots, N - 1$ , by using Taylor theorem we get

$$\begin{aligned} Y_1(t_{n+1}; r) &= Y_1(t_n; r) + \frac{h}{6} F[t_n, Y_1(t_n; r), Y_2(t_n; r)] + \frac{73}{720} h^5 MP^4 + O(h^6), \\ Y_2(t_{n+1}; r) &= Y_2(t_n; r) + \frac{h}{6} G[t_n, Y_1(t_n; r), Y_2(t_n; r)] + \frac{73}{720} h^5 MP^4 + O(h^6), \end{aligned} \tag{16}$$

denote

$$W_n = Y_1(t_n; r) - y_1(t_n; r),$$

$$V_n = Y_2(t_n; r) - y_2(t_n; r).$$

Hence from (15) and (16)

$$W_{n+1} = W_n + \frac{h}{6} \{F[t_n, Y_1(t_n; r), Y_2(t_n; r)] - F[t_n, y_1(t_n; r), y_2(t_n; r)]\} + \frac{73}{720} h^5 MP^4 + O(h^6),$$

$$V_{n+1} = V_n + \frac{h}{6} \{G[t_n, Y_1(t_n; r), Y_2(t_n; r)] - G[t_n, y_1(t_n; r), y_2(t_n; r)]\} + \frac{73}{720} h^5 MP^4 + O(h^6).$$

Then

$$|W_{n+1}| \leq |W_n| + \frac{1}{3} Lh \cdot \max\{|W_n|, |V_n|\} + \frac{73}{720} h^5 MP^4 + O(h^6),$$

$$|V_{n+1}| \leq |V_n| + \frac{1}{3} Lh \cdot \max\{|W_n|, |V_n|\} + \frac{73}{720} h^5 MP^4 + O(h^6),$$

for  $t \in [0, T]$  and  $L > 0$  is a bound for the partial derivatives of  $F$  and  $G$ . Thus by

lemma 4.2

$$|W_n| \leq (1 + \frac{2}{3} Lh)^n |U_0| + (\frac{73}{360} h^5 MP^4 + O(h^6)) \frac{(1 + \frac{2}{3} Lh)^n - 1}{\frac{2}{3} Lh},$$

$$|V_n| \leq (1 + \frac{2}{3} Lh)^n |U_0| + (\frac{73}{360} h^5 MP^4 + O(h^6)) \frac{(1 + \frac{2}{3} Lh)^n - 1}{\frac{2}{3} Lh},$$



where  $|U_0| = |W_0| + |V_0|$ . In particular

$$|W_N| \leq (1 + \frac{2}{3}Lh)^N |U_0| + (\frac{73}{240}h^4MP^4 + O(h^5)) \frac{(1 + \frac{2}{3}Lh)^{\frac{T}{h}} - 1}{L},$$

$$|V_N| \leq (1 + \frac{2}{3}Lh)^N |U_0| + (\frac{73}{240}h^4MP^4 + O(h^5)) \frac{(1 + \frac{2}{3}Lh)^{\frac{T}{h}} - 1}{L}.$$

Since  $W_0 = V_0 = 0$  we obtain

$$|W_N| \leq (\frac{73}{240}MP^4) \frac{e^{\frac{2}{3}LT} - 1}{L} h^4 + O(h^5),$$

$$|V_N| \leq (\frac{73}{240}MP^4) \frac{e^{\frac{2}{3}LT} - 1}{L} h^4 + O(h^5)$$

and if  $h \rightarrow 0$  we get  $W_N \rightarrow 0$  and  $V_N \rightarrow 0$  which completes the proof.

### 5. Examples

**Example 5.1.** Consider the fuzzy initial value problem, [10],

$$\begin{cases} y'(t) = y(t), & t \in I = [0, T], \\ y(0) = (0.75 + 0.25r, 1.125 - 0.125r), & 0 < r \leq 1. \end{cases}$$

By using 4<sup>th</sup> order Runge-Kutta method we have

$$y_1(t_{n+1}; r) = y_1(t_n; r) [1 + h + \frac{h^2}{2} + \frac{h^3}{6} + \frac{h^4}{24}],$$

$$y_2(t_{n+1}; r) = y_2(t_n; r) [1 + h + \frac{h^2}{2} + \frac{h^3}{6} + \frac{h^4}{24}],$$

The exact solution is given by

$$Y_1(t; r) = y_1(0; r) e^t, \quad Y_2(t; r) = y_2(0; r) e^t$$

which at  $t = 1$ ,

$$Y(1; r) = [(0.75 + 0.25r)e, (1.125 - 0.125r)e], 0 < r \leq 1.$$

The exact and approximate solutions are compared and plotted at  $t = 1$  in Figure (5.1).

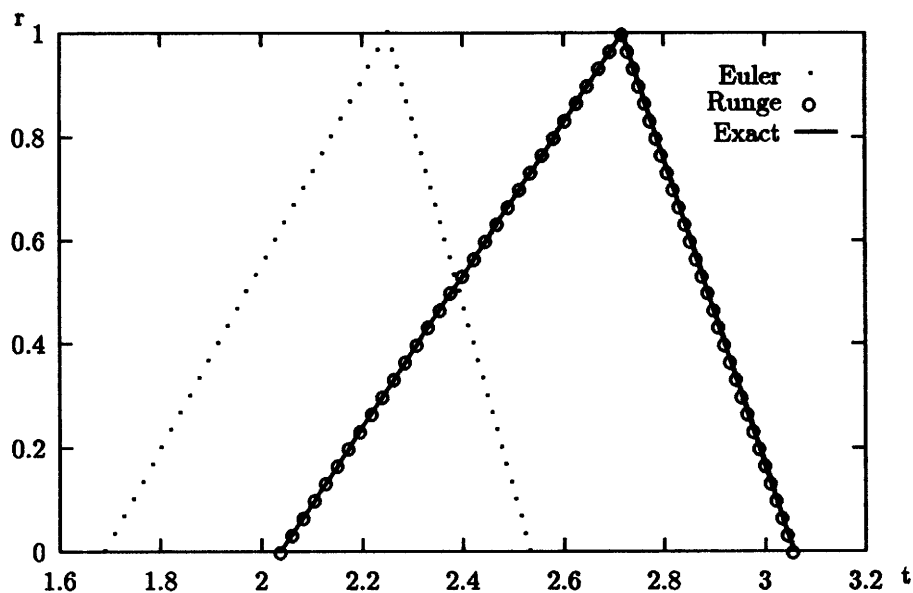


Figure (5.1), ( $h=0.5$ )

**Example 5.2.** Consider the fuzzy initial value problem, [10],

$$\begin{cases} y'(t) = ty(t), [a, b] = [-1, 1], \\ y(-1) = (\sqrt{e} - 0.5(1-r), \sqrt{e} + 0.5(1-r)), 0 < r \leq 1. \end{cases}$$

We separate between two steps.

(a)  $t < 0$ : The parametric form in this case is

$$y'_1(t; r) = ty_2(t; r) \quad , \quad y'_2(t; r) = ty_1(t; r),$$

with the initial conditions given. The unique exact solution is

$$\begin{aligned} Y_1(t; r) &= \frac{A-B}{2} y_2(0; r) + \frac{A+B}{2} y_1(0; r), \\ Y_2(t; r) &= \frac{A+B}{2} y_2(0; r) + \frac{A-B}{2} y_1(0; r), \end{aligned}$$

where  $A = e^{\frac{(t^2-a^2)}{2}}$ ,  $B = \frac{1}{A}$ .

(b)  $t \geq 0$ : The parametric equations are

$$y'_1(t; r) = ty_1(t; r) \quad , \quad y'_2(t; r) = ty_2(t; r),$$

with the initial conditions given. The unique exact solution at  $t > 0$  is

$$Y_1(t; r) = y_1(0; r)e^{\frac{t^2}{2}} \quad , \quad Y_2(t; r) = y_2(0; r)e^{\frac{t^2}{2}} .$$

By using 4<sup>th</sup> order Runge-Kutta method at  $t_n, 0 \leq n \leq N$  we have

$$\begin{aligned} k_{1,1}(t_n; r) &= \min\{t.u \mid u \in [y_1(t_n; r), y_2(t_n; r)]\}, \\ k_{1,2}(t_n; r) &= \max\{t.u \mid u \in [y_1(t_n; r), y_2(t_n; r)]\}, \\ k_{2,1}(t_n; r) &= \min\{(t + \frac{h}{2}).u \mid u \in [z_{1,1}(t_n; r), z_{1,2}(t_n; r)]\}, \\ k_{2,2}(t_n; r) &= \max\{(t + \frac{h}{2}).u \mid u \in [z_{1,1}(t_n; r), z_{1,2}(t_n; r)]\}, \\ k_{3,1}(t_n; r) &= \min\{(t + \frac{h}{2}).u \mid u \in [z_{2,1}(t_n; r), z_{2,2}(t_n; r)]\}, \\ k_{3,2}(t_n; r) &= \max\{(t + \frac{h}{2}).u \mid u \in [z_{2,1}(t_n; r), z_{2,2}(t_n; r)]\}, \\ k_{4,1}(t_n; r) &= \min\{(t + h).u \mid u \in [z_{3,1}(t_n; r), z_{3,2}(t_n; r)]\}, \\ k_{4,2}(t_n; r) &= \max\{(t + h).u \mid u \in [z_{3,1}(t_n; r), z_{3,2}(t_n; r)]\}. \end{aligned}$$

Where

$$\begin{aligned} z_{1,1}(t_n; r) &= y_1(t_n; r) + \frac{h}{2}k_{1,1}(t_n; r) \quad , \quad z_{1,2}(t_n; r) = y_2(t_n; r) + \frac{h}{2}k_{1,2}(t_n; r), \\ z_{2,1}(t_n; r) &= y_1(t_n; r) + \frac{h}{2}k_{2,1}(t_n; r) \quad , \quad z_{2,2}(t_n; r) = y_2(t_n; r) + \frac{h}{2}k_{2,2}(t_n; r), \\ z_{3,1}(t_n; r) &= y_1(t_n; r) + hk_{3,1}(t_n; r) \quad , \quad z_{3,2}(t_n; r) = y_1(t_n; r) + hk_{3,2}(t_n; r). \end{aligned}$$

By considering  $t > 0$  and  $t < 0$ , the above minimizing and maximizing problems can be solved by GAMS software. The exact and approximate solutions are compared and plotted in Fig.(5.2).

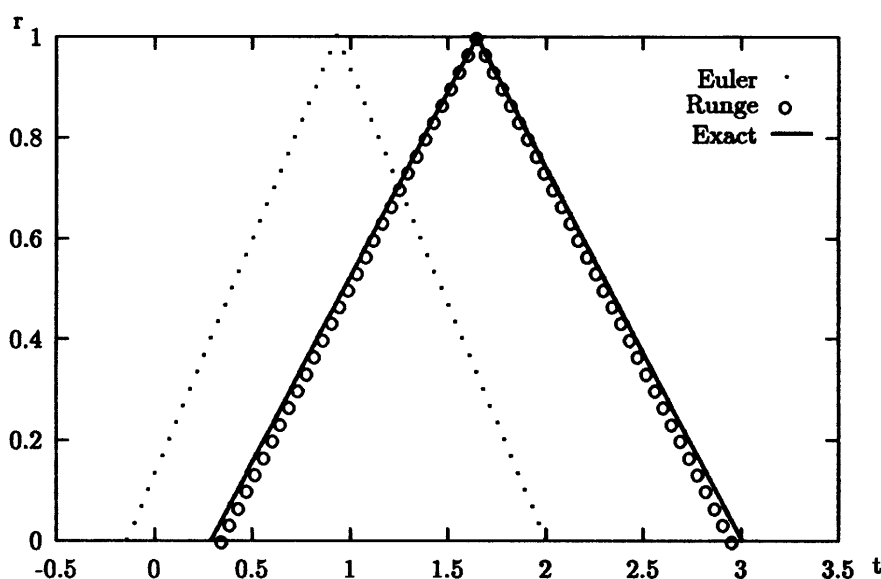


Figure (5.2), ( $h = 0.4$ )

**Example 5.3.** Consider the fuzzy initial value problem

$$y'(t) = c_1 y^2(t) + c_2, \quad y(0) = 0,$$

where  $c_i > 0$ , for  $i = 1, 2$  are triangular fuzzy numbers, [11].

The exact solution is given by

$$Y_1(t; r) = l_1(r) \tan(\omega_1(r)t),$$

$$Y_2(t; r) = l_2(r) \tan(\omega_2(r)t),$$

with

$$l_1(r) = \sqrt{c_{2,1}(r)/c_{1,1}(r)}, \quad l_2(r) = \sqrt{c_{2,2}(r)/c_{1,2}(r)},$$

$$\omega_1(r) = \sqrt{c_{1,1}(r)/c_{2,1}(r)}, \quad \omega_2(r) = \sqrt{c_{1,2}(r)/c_{2,2}(r)},$$

where

$$[c_1]_r = [c_{1,1}(r), c_{1,2}(r)] \quad \text{and} \quad [k_2]_r = [c_{2,1}(r), c_{2,2}(r)]$$

and

$$c_{1,1}(r) = 0.5 + 0.5r, \quad c_{1,2}(r) = 1.5 - 0.5r, \quad c_{2,1}(r) = 0.75 + 0.25r, \quad c_{2,2}(r) = 1.25 - 0.25r.$$

The  $r$ -level sets of  $y'(t)$  are

$$Y'_1(t; r) = c_{2,1}(r) \sec^2(\omega_1(r)t),$$

$$Y'_2(t; r) = c_{2,2}(r) \sec^2(\omega_2(r)t),$$

which defines a fuzzy number. We have

$$f_1(t, y; r) = \min\{c_1 u^2 + c_2 \mid u \in [y_1(t; r), y_2(t; r)], c_1 \in [c_{1,1}(r), c_{1,2}(r)], c_2 \in [c_{2,1}(r), c_{2,2}(r)]\},$$

$$f_2(t, y; r) = \max\{c_1 u^2 + c_2 \mid u \in [y_1(t; r), y_2(t; r)], c_1 \in [c_{1,1}(r), c_{1,2}(r)], c_2 \in [c_{2,1}(r), c_{2,2}(r)]\},$$

By using 4<sup>th</sup> order Runge-Kutta method at  $t_n, 0 \leq n \leq N$  we have

$$k_{1,1}(t_n; r) = c_{1,1}(r) \cdot y_1^2(t_n; r) + c_{2,1}(r),$$

$$k_{1,2}(t_n; r) = c_{1,2}(r) \cdot y_2^2(t_n; r) + c_{2,2}(r),$$

$$k_{2,1}(t_n; r) = c_{1,1}(r) \cdot z_{1,1}^2(t_n; r) + c_{2,1}(r),$$

$$k_{2,2}(t_n; r) = c_{1,2}(r) \cdot z_{1,2}^2(t_n; r) + c_{2,2}(r),$$

$$k_{3,1}(t_n; r) = c_{1,1}(r) \cdot z_{2,1}^2(t_n; r) + c_{2,1}(r),$$

$$k_{3,2}(t_n; r) = c_{1,2}(r) \cdot z_{2,2}^2(t_n; r) + c_{2,2}(r),$$

$$k_{4,1}(t_n; r) = c_{1,1}(r) \cdot z_{3,1}^2(t_n; r) + c_{2,1}(r),$$

$$k_{4,2}(t_n; r) = c_{1,2}(r) \cdot z_{3,2}^2(t_n; r) + c_{2,2}(r).$$

where

$$\begin{aligned} z_{1,1}(t_n; r) &= y_1(t_n; r) + \frac{h}{2} k_{1,1}(t_n; r) & , & \quad z_{1,2}(t_n; r) = y_2(t_n; r) + \frac{h}{2} k_{1,2}(t_n; r), \\ z_{2,1}(t_n; r) &= y_1(t_n; r) + \frac{h}{2} k_{2,1}(t_n; r) & , & \quad z_{2,2}(t_n; r) = y_2(t_n; r) + \frac{h}{2} k_{2,2}(t_n; r), \\ z_{3,1}(t_n; r) &= y_1(t_n; r) + hk_{3,1}(t_n; r) & , & \quad z_{3,2}(t_n; r) = y_1(t_n; r) + hk_{3,2}(t_n; r). \end{aligned}$$

There are two nonlinear programming problems which can be solved by GAMS software. Thus the suggested 4<sup>th</sup> order Runge-Kutta method of this paper can be used. The exact and approximate solutions are shown in Figure (5.3) at  $t = 1$ .

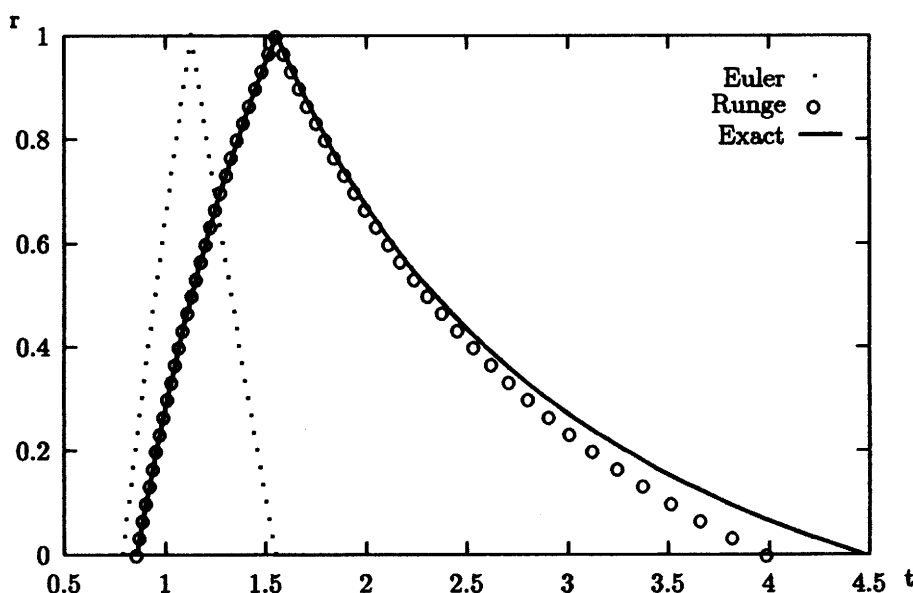


Figure (5.3), ( $h = 0.5$ )

### 6. Conclusion

We note that the convergence order of the Euler method in [10] is  $O(h)$ . It is shown that in proposed method, the convergence order is  $O(h^4)$  and the comparison of solutions of examples (1) ,(2) of this paper and [10] shows that these solutions are better for these examples.

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