Preconditioned conjugate gradient method for solution of linear operator integral equation of the first kind

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Abstract

In this artical, the solution of linear operator equations of first kind is reconstructed with definition of an optimal method .This method is a combination of optimization theorems and conjugate gradient iteration preconditioned.Idea of preconditioning over large linear systems is transformed on equivalent problem with a small condition number,which increases the rate of convergence. Only the case of positive-definite ,symmetric operators on an inner product space is considered .This method can be applied also on compact operators.Moreover, several numerical examples are given.

Key words. *Integral equation,Preconditioned Conjugate Gradient,Toeplitz systems,Sparse matrix.*
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1-Introduction

The main difficulty in solving a Fredholm integral equation of the first kind

(1) $$Kz(s)=\int_a^b k(s,t)z(t)dt=g(s)\ ,\ c\le s\le d$$

arises from the instability of the (generalized)inverse operator. Tikhonov [16] proposed to damp out such oscillations and **regularize** the solution process by taking as an approximate solution to the function z that reduces,

(2) $$\int_0^1 (K(Z(s)-g(s))ds+\alpha\int_0^1 [p(s)z^2(t)+q(s)(\frac{dz(s)}{ds})^2 ds$$

where p and q are strictly positive functions and α is a positive parameter .This idea may be phrased abstractly as the problem of minimizating of the functional

(3) $$F_\alpha(z)=\|Kz-g\|^2+\alpha\|z\|$$

where K is a compact operator from a real Hilbert space H_1 into a real Hilbert space H_2. A family of Tikhonov-type regularization methods for solving(1) has proposed, that is

(4) $$F_\alpha(z)=\|Kz-g\|^2+\alpha\|z^{(n)}\|^2\ ,n=1,2,...$$

Where K is differential operator and n order of derivative. In [15], Marti has shown that Sobolev spaces may serve as a tool for a partial regularization of equations of type (1). Where a Sobolev space $H^m(0,1)$, $m\geq 0$ is defined as the completion of the set $C^m(0,1)$(bounded continuous and m-times bounded continuously differentiable real function on (0,1)) with respect to the Sobolev norm given by:

$\| f \|_m = (\sum_{i=0}^{m} \int_0^1 f^{(i)^2}(t)dt)^{\frac{1}{2}}$, $f \in C^m(0,1)$. Or the norm $\|\cdot\|_m$ of these Hilbert spaces generated by the inner product $(,)_m$ defined by:

$(f,g)_m = (\sum_{i=0}^{m} \int_0^1 f^{(i)}(t) g^{(i)}(t)dt)$, $f \in C^m(0,1)$. Let W_n be a subset of $H^o(0,1)$ with finite dimension n, and $\langle v_i \rangle$ be a basis of W_n. Now we have (1) and we defined :

(5) $$\| Kz_n - g \|_m^2 = (Kz-g, Kz-g)_m = (Kz,Kz)_m - 2(Kz,g)_m + (g,g)_m$$

Since $H^o(0,1)$ is a compact space ,so $z_n = \sum_{i=1}^{n} x_i v_i$ globally converges to z [6].Now we have:

(6) $$\| Kz_n - g \|_m^2 = \sum_{i,j=1}^{n} x_i x_j (Kv_i, Kv_j)_m - 2\sum_{i=1}^{n} x_i (Kv_i, g)_m + (g,g)_m$$

defining $B=[(Kv_i ,Kv_j)_m]$, $W=[(Kv_i , g)_m]$, $M=[(v_i , v_j)_m]$ and substituting in (6)we obtain:

(7) $$\| Kz_n - g \|_m^2 = X^T BX - 2W^T X + (g,g)_m$$

similarly we have

(8) $$\| z_n \|_m^2 = \sum_{i,j=1}^{n} x_i x_j (v_i, v_j) = X^T MX$$

So by (7),(8) and substitution in (3) we have

(9) $$F_\alpha(X) = X^T BX - 2W^T X + (g,g) + \alpha X^T MX \quad (\alpha>0)$$

2-Relation between Integral Operator and optimization

A general quadratic function can be written as

(10) $$F_\alpha(X) = X^T G^\alpha X - b^T X + C \quad (\alpha>0)$$

(11)
$$\underset{\alpha}{F}(X)=\frac{1}{2}(X-X^{*})^{T}G^{\alpha}(X-X^{*})+C^{*}$$

where $G^{\alpha}X^{*}=-b$, $C^{*}=C-\frac{1}{2}X^{*T}GX^{*}$. G^{α} (or the Hessian matrix)is symmetric and maps differences in position into differences gradient(that is if $\underset{\alpha}{\overset{*}{g}}=\nabla \underset{\alpha}{F}(X^{*})$, $\underset{\alpha}{\overset{**}{g}}=\nabla \underset{\alpha}{F}(X^{**})$ then $\underset{\alpha}{\overset{*}{g}}-\underset{\alpha}{\overset{**}{g}}=G^{\alpha}(X^{*}-X^{**})$ .Sufficient conditions which imply that x^{*} is a local minimizer of (10) are as follows .

Theorem 1) Sufficient conditions for a strict and isolated local minimizer x^{*} are that $\underset{\alpha}{\overset{*}{g}}=0$ holds and that $(G^{\alpha})^{*}$ is positive definite ,that is $S^{T}(G^{\alpha})^{*}S>0$ for $\forall S\neq 0$.
Proof :In [13].

Other main difficulty is solving equation (10), to determine exact bounds on regularization parameter (that is $\alpha>0$) . If α is equal to zero in gradient (9) and $\underset{T}{x}$ denote exact solution , then we have $B\underset{T}{x}=\underset{T}{W}$, and we have,

(12)
$$B(x-\underset{T}{x})+\alpha Mx=W-\underset{T}{W}=\delta W$$.

Assume that inverse of B exists and it is bounded. It is shown in [13] that if we are permitted an error $\underset{1}{\epsilon}$ in the sense of $\|Kx-y\|\leq\epsilon_{1}$,then we must have in $J(x)=\|Kx-y\|+\alpha\phi(x)$, $\alpha\leq\frac{\epsilon_{1}}{\phi(x)}$, and by attention to (8), $\alpha\leq\frac{\epsilon_{1}}{x^{T}Mx}$ (Let $\phi(x)=x^{T}Mx$ satisfy conditions in [13]). So we have

(13)
$$\frac{\underset{1}{\epsilon}}{x^{T}Mx}\geq\alpha\geq 0 .$$

Theorem 2) If $\hat{\alpha}$ exists and satisfies in(13), and $B+\hat{\alpha}M$ is positive definite then a minimizer x^{*} for (9) exists and x^{*} is a strict and isolated local minimizer.
Proof : Use theorem (1).

<u>Remark</u>: It seems that with identifying bounds for α ,we can find out the solutions of an integral equation with compact linear operator ,but it is difficult to find out the

al α . Also ,the different values of α have high effects on solutions, hence, it l be assumed that K is a positive definitive and symmetric, because in this case em can be stable(see theorem 3).

orem 3) let X and Y be subspaces of an inner product space Z ,and let $K:X\to Y$ be ar ,positive-definite ,symmetric operator .If x satisfies Kx=y ,then it minimizes the onal

$$q(x)=(Kx,x)-2(y,x)$$

f : [13].

other form is assumed to have a coefficient matrix A that is real ,symmetric and ve definite ,the solution of this system is equivalent to the minimization of the ion

$$f(x)=\frac{1}{2}x^{T}Ax-b^{T}x \qquad x\in\mathbb{R}^{n}\ .$$

e unique solution x^* of Ax=b is also the unique minimizer of f(x) as x varies over he finite range of X. Let us choose a set of linearly independent ents v_{ni} $i=1,2,\ldots,n$, $n=1,2,\ldots$ in X

d minimize q(x) over $X_n=span(v_{ni})$. Then write $z_n=\sum_{i=1}^{n}x_i v_{ni}$ and have

$$z_n)=\sum_{i=1}^{n}\sum_{j=1}^{n}x_i x_j(Lv_{ni},v_{nj})-2\sum_{i=1}^{n}x_i(v_{ni},y)$$ then we have

$$q(z_n)=F(x)=x^{T}Gx-2x^{T}b\ .$$

is is a quadratic function ,but probably it is an improperly posed problem .Consider), if $\nabla F(x)=0$, then the linear system to be solved,

$$Gx=b \quad ,$$

)

assumed to have a coefficient matrix G that is real and positive definite . The ution of this system is equivalent to the minimization of function (16) , where $x\in R^n$. well-known iteration method for finding a minimum for a nonlinear function is the thod of steepest descent [5]. For minimizing F(x) by this method , assume that an tial guess x_0 is given. Choose a path and search for a new minimum on it, by checking nditions of theorem 2, it must be $\nabla F(x_0)<0$. We move along the direction $^{0)}=\nabla F(x_0)=g(x_0)=g_0=b-Gx_0$,then solve the one dimensional minimization problem $nF(x_0+\gamma g_0)$, calling the solution γ_1 (by linear search methods [18]).

Using it ,define the new iteration $x_{k+1}=x_k+\gamma_k g_k$ $k=0,1,2,\ldots$. The method of steepest scent will converge [5,9] but the convergence is generally quite slow . The optimal local strategy of using a direction of fastest descent is not a good strategy for finding an optimal direction for finding the global minimum .In comparison ,the conjugate gradient method(CG) will be more rapid, it will take no more than n iterations , assuming there are no rounding errors.

3 -Preconditioned Conjugate Gradient Method (PCG)

A particular way of obtaining quadratic termination is to invoke the concept of the conjugacy a set of non-zero vectors $s^{(1)},\dots,s^{(n)}$ to a given positive definite matrix G . This is property that

(18) $$S^{(i)^T} G S^{(j)}=0 \quad , \quad \forall\ i\neq j$$

Theorem 4) A conjugate direction method terminates for a quadratic function in at most n exact line search ,and each $x^{(k+1)}$is the minimizer in subspace generated by $x^{(1)}$and the directions $S^{(1)},S^{(2)},\dots,S^{(k)}$(that is the set of points $\{\ x|\ x=x^{(1)}+\sum_{j=1}^{k}\gamma_j S^{(j)} \quad ,\forall\gamma_j\ \}$) Proof: In[9].

An equivalent geometric definition can be given by introducing a new inner product and norm for $R^n: (x,y)_A=x^TAy \quad ,\|x\|_A=\sqrt{(x,x)_A}=\sqrt{x^TAx} \quad x\in R^n$.

Given a set of conjugate directions $\{S^{(1)},\dots,S^{(n)}\}$ it is straightforward to solve Gx=b. Let $x^*=\gamma_1 S^{(1)}+\gamma_2 S^{(2)}+\dots+\gamma_n S^{(n)}$ using(18),

(19) $$\gamma_k=\frac{S^{(k)^T}\ G\ x^*}{S^{(k)^T}\ G\ S^{(k)}} \qquad k=1,2,\dots,n$$

We use this formula forx^* to introduce the conjugate direction method . let $x^0=0$,

(20) $$x^{(k)}=\gamma_1 S^{(1)}+\dots+\gamma_k S^{(k)} \qquad 1\le k\le n.$$

Let $r^{(k)}=b-Gx^{(k)}=-\nabla F(x^{(k)})$ the residual of $x^{(k)}$in Ax=b. Obviously $r^{(0)}=b$ and

(21) $$x^{(k)}=x^{(k-1)}+\gamma_k S^{(k)} \quad , \quad r^{(k)}=r^{(k-1)}-\gamma_k GS^{(k)} \qquad k=1,\dots,n$$

For k=n ,we have $x_n=x^*$, $r^{(n)}=0$ and x_k may equal x^* with a smaller value of k. To generate the directions different ways can be used, in [9,10] simple method is using steepest desent method direction. Since $x^{(0)}=0$,choose the first direction $S^{(1)}$, as follows: $S^{(1)}=-\nabla F^{(\alpha)}(x^0)=r^0=b$. An inductive construction is given for the remaining directions.

Assume $x^{(1)},\dots,x^{(k)}$ have been generated, along with the conjugate directions $S^{(1)},S^{(2)},\dots,S^{(k)}$. A new direction $S^{(k+1)}$must be chosen ,assume $x^{(k)}\neq x^*$ thus $r^{(k)}\neq 0$, otherwise, we would have the solution x^* and there would be no point to continue . By theorem (3),$r^{(k)}$is orthogonal to $M=(S^{(1)},...,S^{(k)})$, $r^{(k)}$does not belong to M . Putting $S^{(k+1)}=r^{(k)}+\beta_{k+1}S^{(k)}$,then the condition $S^{(k)T}GS^{(k+1)}=0$ implies

(22) $$\beta_{k+1}=-\frac{S^{(k)T}G\,r^k}{S^{(k)T}G\,S^k}$$

The error analysis of the CG method is based on the following optimization result.

Theorem 5) The sequence $(x^{(k)})$ of the CG method satisfies in

(23) $$\| x^*-x^{(k)}\|_G=\min_{deg(q(G))<k}\| x^*-q(G)\,b\|_G$$

Proof : [14] ,(where $q(\lambda)$ is a polynomial ,for example $q(\lambda)=a_0+a_1\lambda+a_2\lambda^2$).

Now, if the eigenvalues of G be denoted by $0<\sigma_1<\sigma_2<...<\sigma_n$ and let $u_1,...,u_n$ denote acorresponding orthonormal basis of eigenvectors . Using this basis, we can write

$x^*=\sum_{j=1}^{n}a_ju_j$, $b=Gx^*=\sum_{j=1}^{n}a_j\,\sigma_j\,u_j$ then

(24) $$q(G)b=\sum_{j=1}^{n}a_j\,\sigma_j\,q(\sigma_j)u_j$$

with attention to (23) ,(24) we have :

(25) $$\| x^*-q(G)b\|_G=[\sum_{j=1}^{n}a_j^2\,\sigma_j\,(1-\sigma_j\,q(\sigma_j)^2)]^{\frac{1}{2}}$$

In [9] we can find better known bounds :

(26) $$\frac{\| x^*-x^{(k)}\|_G}{\| x^*\|_G}\le 2\left(\frac{1-\sqrt{c}}{1+\sqrt{c}}\right)^k$$

where $c=\sigma_1\sigma_n^{-1}$, c is the condition number of G .The bound (26) implies that the CG methods can converge quit slowly. To increase the rate of convergence ,or at least to grant a rapid rate of convergence, the problem (17) is transformed to an equivalent problem with smaller condition number. The bounds in (26), will be smaller and one expects that the iteration will converge more rapidly.

If we have (17) ,then matrix Q is chosen such that Q is nonsingular, so that we transform (17) by $(Q^{-1}GQ^{-T})(Q^Tx)=Q^{-1}b$, converted to $\bar{G}\bar{x}=\bar{b}$ where $\bar{G}=Q^{-1}GQ^{-T}$, $\bar{x}=Q^Tx$, $\bar{b}=Q^{-1}b$, then $cond(\bar{G})_2$ is smaller than $cond(G)_2$ (see appendix).

Finding Q requires a careful analysis of the original problem (1)and understanding the structure of G in order to choose Q.If$G=Q\,\bar{G}\,Q^T$ with $\bar{G}$ to be chosen with eigenvalues near 1 in magnitude, for example ,if $\bar{G}$ is about the identity I ,then $G=Q\,Q^T$

Approximate Cholesky factors are used in preconditioners in some cases [11].The

Approximate Cholesky factors are used in preconditioners in some cases [11].The success of the pre-conditioned conjugate gradient method depends upon the choice of the preconditioner .Assume matrix M has given , following (17) $M^{-1}Gx=M^{-1}b$.So we have [9]

$$\frac{\| x^{n}-x^{*}\|_{G}}{\| x^{0}-x^{*}\|_{G}} \leq 4\left(\frac{\sqrt{c}-1}{\sqrt{c}+1}\right)^{2n} \tag{27}$$

It has been shown that if $K=M^{-1}G$ has p distinct eigenvalues then CG method converges in at most p iterations .If we consider the system of equations (17) where G is a symmetric positive definite matrix ,then we re-write G=M-N , where M is symmetric and positive definite, we assume that a given vector ,say, d can solve the system Mz=d easily. So the preconditioned conjugate gradient (PCG) method proceeds as follows:

Algorithm

Generate G,M,b

$k=0;\quad x_0=0\quad ;r^{(0)}=b$

while $(r^{(k)}\neq 0)$

solve $MZ^{(k)}=b-Gx^{(k)}=r^{(k)}$

$k=k+1$

if $k=1$

$S^{(1)}=Z^{(0)}$

else

$\beta_k=(r^{T(k-1)}Z^{(k-1)})/(r^{T(k-2)}Z^{(k-2)})$

$S^{(k)}=Z^{(k-1)}+\beta_k S^{(k-1)}$

end

$\gamma_k=(r^{T(k-1)}Z^{(k-1)})/(S^{T(k)}GS^{(k)})$

$x^{(k)}=x^{(k-1)}+\gamma_k S^{(k)}$

$r^{(k)}=r^{(k-1)}-\gamma_k GS^{(k)}$

end

$x=x^{(k)}$

Matrix M in the algorithm is the preconditioning matrix. In many integral equations, matrix G of the following structure arises(or sparse matrices representation of discritization of Integral operators [1])

$$\begin{pmatrix} G_1 & 0 & \cdots & o & B_1 \\ 0 & G_2 & \cdots & 0 & B_2 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & G_r & B_r \\ B_1 & B_2 & \cdots & B_r & Q \end{pmatrix}$$

In this case matrix G is symmetric and positive definite so $G_i(n_i, n_i)$ is also positive definite and also Q is a $p\times p$ matrix, construction of conditioner should be easy ,but in the case of non-symmetric we can use other hods [1,2,8,12,17]. Suppose we wish to solve the system (17) where

$$\begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_r \\ \xi \end{pmatrix}, \quad b=\begin{pmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ b_r \\ c \end{pmatrix}$$

One possibility is to apply the CG method to (17) with

$$M=\begin{pmatrix} G_1 & 0 & \cdots & o & 0 \\ 0 & G_2 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & G_r & B_r \\ 0 & 0 & \cdots & 0 & Q \end{pmatrix}$$

Note that the matrix $M^{-1}(G-M)$ is 2-cyclic .We obtain envalues of $M^{-1}(G-M)$ with matrix equation $M^{-1}(G-M)U=\lambda u$. Then if $U=(u_1,u_2,\ldots,u_r,v)^T$ sy manipulation shows

$$\sum_{i=1}^{r} B_i^T G_i v = \lambda^2 Qv \qquad \text{3)}$$

nstead of solving (17) directly , we could eliminate $x_1,x_2,\ldots,x_r$ and solve for ξ. This leads the system

$$(Q-\sum_{i=1}^{r} B_i^T G_i^{-1} B_i)\xi = c-\sum_{i=1}^{r} B_i^T G_i^{-1} b_i \qquad \text{9)}$$

f Q is a $p\times p$ matrix then we can apply the CG method to (29) .We will obtain, the lution in at most p iterations .The matrix $A=Q-\sum_{i=1}^{r} B_i^T G_i^{-1} B_i$ is the Schur complement of in G and hence A is symmetric and positive definite. Associated with (29) , we can oose a preconditioner $\tilde{M}$.For example $\tilde{M}=I$ or $\tilde{M}=Q$. Note that if $\tilde{M}\xi=\tilde{N}\xi+b$, then the nvergence properties of the algorithm are determined by the eigenvalues of $\tilde{M}^{-1}\tilde{N}$

Let $\tilde{M}^{-1}\tilde{N}w=\gamma w$. Then if $\tilde{M}=Q$, we have

(30) $$\sum_{i=1}^{r} B_i^T G_i^{-1} B_i w=\gamma Q w$$

Hence the eigenvalues of (28) are the squares of eigenvalues of (30). The rate of convergence of the two procedures are essentially the same, because we are able to eliminate half of the numerical operations. On the other hand Toeplitz matrix arises in many applications of integral equations. A Toeplitz matrix is constant along its diagonals and thus , in the symmetric case , is determined by the n elements of the first row , $a_0 a_1 ,..., a_{n-1}$

(31) $$G=\begin{pmatrix} a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} \\ a_1 & a_0 & a_1 & . & a_{n-2} \\ a_2 & . & . & . & . \\ . & . & . & . & . \\ a_{n-1} & . & a_2 & a_1 & a_0 \end{pmatrix}$$

A circulan matrix , which is special case of a Toeplitz matrix, is determined by only the first $\frac{n}{2}+1$ elements of first row if n is even . Each successive row contains the elements of row above shifted one to the right , with the last element wrapped around to become the first ,i.e.;

$$\begin{pmatrix} c_0 & c_1 & \cdots & c_2 & c_1 \\ c_1 & c_0 & c_1 & . & c_2 \\ c_2 & c_1 & . & . & . \\ . & . & . & . & c_1 \\ c_0 & . & c_2 & c_1 & c_0 \end{pmatrix}$$

Since the eigenvectors of a circulant matrix are given by successive powers of the n th roots of unity , systems with circulant coefficient matrix are amenable to solution by the FFT in O(nlog n) operations .Usually a preconditioner (31) can be obtained as follows:

$$S=\begin{pmatrix} a_0 & a_1 & \cdots & a_2 & a_1 \\ a_1 & a_0 & a_1 & \cdot & a_2 \\ a_2 & \cdot & & \cdot & \\ \cdot & \cdot & & \cdot & \cdot \\ a_1 & \cdot & \cdot & a_1 & a_0 \end{pmatrix}$$

The matrix S is circulant preconditioner for G [4,7].Now , we are in position to construct circulant matrix in linear operator integral equation by using periodic splines [6,7].

4-Numerical Experimnts

Consider $\int_a^a k(x-y)f(x)=g(x), a\leq\infty$, let f be approximated by

$f_M(x)=\sum_{j=0}^{M-1}\alpha_j B_j(h;x)$, $B_j(h;x)=\frac{1}{6}\sum_{k=0}^{4}(-1)^k\binom{4}{k}(\frac{x}{h}-j-k)_+^3$ where $x_+=max(0,x)$ and $B_j(h;x)$ are

periodic cubic cardinal B-splines with period T=M h and knot spacing h. M is the

number of B-splines. The vector $\alpha=(\alpha_0\alpha_1, \ldots, \alpha_{M-1})^T$ of unknow coefficients is to be

determined. Since $B_0(h;x)$ is periodic on (0,T) ,it has the Fourier series

$B_0(h;x)=\frac{1}{T}\sum_{q=-\infty}^{\infty}\hat{B}_0 e^{(i w_q x)}$ where $w_q=[\frac{2\pi}{T}]q$ and $\hat{B}_{0q}=\int_0^T B_0(h;x)e^{(-i w_q x)}dx=h\left(\frac{\sin(\frac{h w_q}{2})}{\frac{h w_q}{2}}\right)^4$ so

$\hat{B}_{jq}=\hat{B}_{0q}\, e^{(-iw_q jh)}$, $q=\ldots,-1,0,1,\ldots$, $j=0,\ldots,M-1$ the spline in equation $f_M(x)=\sum_{j=0}^{M-1}\alpha_j B_j(h;x)$ has the

Fourier series $f_M(x)=\frac{1}{T}\sum_{q=-\infty}^{+\infty}\hat{f}_{Mq}\exp(iw_q x)$ with Fourier cofficients $\hat{f}_{Mq}=\sum_{j=0}^{M-1}\alpha_j\hat{B}_{jq}$. Moreover

functions k and g are approximated on (0,T) by:

$k_N(x)=\frac{1}{N}\sum_{q=0}^{N-1}\hat{k}_{N,q}\exp(i\, w_q x)$, $g_N(x)=\frac{1}{N}\sum_{q=0}^{N-1}\hat{g}_{N,q}\exp(i\, w_q x)$

where $\hat{k}_{Nq}=\sum_{n=0}^{N-1}\hat{k}_n \exp(i w_q x_n)$, $\hat{g}_{Nq}=\sum_{n=0}^{N-1}\hat{g}_n \exp(i w_q x_n)$,$q=0,1,...,N-1$ with $g(x)_n=g_n=g(x)_{Nn}$,$k(x)_n=k_n=k(x)_{Nn}$

and $w_q=\frac{2\pi}{Nh}q$. So we have $k_N*f_M=g_N$. Using least square method and Plancherel's theorem we have $\|k_N*f_M-g_N\|_2^2=\frac{T}{N^2}\sum_{q=0}^{N-1}|\hat{k}_{Nq}\hat{f}_{Mq}-\hat{g}_{Nq}|^2$,with minimizing this functional we have the system $A\alpha=b$ where $A=W^H \hat{P}^2 W$, $b=W^H\hat{P}^2$ and $W=\hat{K}\hat{B}^T$

$\hat{B}=(\hat{B}_{jq})_{jq=0}^{M-1,N-1}$, $\hat{K}=(\hat{k}_{jq}\delta_{qr})_{jqr=0}^{M-1,N-1}$, $\hat{P}=(\frac{\sqrt{T}}{N}\delta_{qr})_{qr=0}^{N-1}$ and $\delta_{q,r}=\begin{cases}1 & q=r\\ 0 & q\neq r\end{cases}$. A is a Toeplitz matrix because $a_{rs}=\sum_{q=0}^{N-1}a_q\exp(\frac{2\pi}{M}iq(r-s))$, $a_q=\frac{T}{N^2}|\hat{k}_{Nq}\hat{B}_{0q}|^2$. If kernel is not smooth then the system $A\alpha=b$ is ill-posed ,so for example let A be a Hilbert-Toeplitz matrix ,as follow:

i) $A_1=((i-j)^2)_{i,j=0}^{M-1}$

ii) $A_2=(e^{|i-j|^2})_{i,j=0}^{M-1}$

Now C and S represent ,resectively Tony chan's[4],circulant preconditioner, and S Strang[3] preconditioner ,then matrices C ,S whose entries are given by

$c_i=\frac{i\,a_{i-M}+(M-i)a_i}{M}$ $i=0,...,(M-1)$ and $S=\sum_{j=0}^{M-1}(\frac{1}{n}\sum_{p-q\equiv j\,(\mathrm{mod}\,n)}a_{pq})Q^j$ where

$$Q=\begin{pmatrix}0 & ... & & 1\\ 1 & 0 & ... & 0\\ 0 & 1 & ... & \\ & & ... & \\ 0 & ... & 1 & 0\end{pmatrix}$$,$C=circul(c_0, c_1,..., c_{M-1})$, $S=circul(s_0 s_0, s_1,..., s_{M-1})$, $A=(a_{ij}=a_{i-j})$.Tables 1 and 2 showing that $cond(A_1)>>cond(S^{-1}A_1)\geq cond(C^{-1}A_1)$ and $cond(A_2)>>cond(S^{-1}A_2)\geq cond(C^{-1}A_2)$.

Table 1

M	$Cond(A)_1$	$Cond(S^{-1}A)_1$	$Cond(C^{-1}A)_1$
3	0.9E01	0.2E01	0.1E01
10	1.1E08	0.6E01	0.2E01
20	8.2E18	0.8E01	0.3E01
100	4.7E19	1.1E01	0.4E01

Table 2

M	$Cond(A)_2$	$Cond(S^{-1}A)_2$	$Cond(C^{-1}A)_2$
3	7.6E01	0.1E01	0.1E01
10	1.1E35	2.1E00	1.3E00
20	$\infty > 1.0E10($	4.3E00	2.7E00

5- Conclusion

We have shown in this paper the difficulty of finding optimal value of regularization parameter α, in spite of having bound for it . For finding α we have to solve large system of equations with large condition numbers. By using PCG method we don't need to find optimal value of α. Constructing special preconditioner we obtain a system of equations with a small condition number .We can summarize some advantages of our algorithm as follows:

(i) Reduction of the order of operations.

(ii) Increasing speed and accuracy of algorithm and saving memories.

(iii) Transform to parallel algorithms.

This paper is only a preliminary study with many open problems.

Appendix

If function F is the generating function of the matrices $A_n(F)$ (i.e. $F(x) = e^{|x|}$ or $x^2 \ldots$) , then theorem 6 is satisfied.

Theorem 6) Let function F belong to Banach space of all 2π-periodic continues real - valued functions ,we have :

(i) Then $\sigma(A_n(F)) \subseteq [F_{min}, F_{max}]$ or $\|A_n(F)\| \leq \|F\|_\infty$ (where σ is spectrum of Toeplitz matrix).

(ii) Let $C_n(F)$ be the n-by-n circulant preconditioner of $A_n(F)$ then $C_n(F)$ is the minimizer of $\|B_n - A_n(F)\|_F$ over all n-by-n circulent matrices B_n .

(iii) Then spectrum of $A_n(F) - C_n(F)$ is clustered around zero (if F is in the Wiener class).

(iv) Then spectrum of $C_n^{-1}(F)A_n(F)$ is clustered around one (if F is in the Wiener class).

(Definition : function F is in the Wiener class if its Fourier coefficients are absolutely summable ,i.e.: $\sum_{k=-\infty}^{\infty} |a_k(F)| < \infty$.)Proof [3,4].

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