ON THE SINGULAR SETS OF A MODULE II

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Throughout this note, A and B wil denote a (non-trivial) commutative Noetherian ring with a multiplicative identity element and M will denote a non-zero finitely generated A-module.

For every non-negative integer k, the set

$S^*_k(M) = \{p \in Spec(A)$ depth $M_p + \dim A/p \leq k\}$

is called the **singular set of M** with respect to k.

It is known that when the ring A is homomorphic image of a biequidimensional regular ring, then the singular sets of M are all closed in the Zariski topology on Spec(A) (see[3; ch. IV, 5]).

A development of this famous theorem has been recently shown in the sence that if A is a homomorphic image of a biequidimensional Gorenstein ring, the singular sets of M are still closed (See[2]).

The purpose of this article is to show that if B homomorphic image of a Cohen-Macaulay local r then $S^*_k(N)$ is closed, for every finitely gener B-module N.

First we prove some preliminary lemmas propositions which help us to conclude the subseq main theorem. From now on, A will denot Cohen-Macaulay local ring with the unique max ideal m, and $\hat{A}$ (respectively $\hat{M}$) will denote the m completion of A (respectively M).

1. Proposition. Let $\phi : A \to \hat{A}$ be the na homomorphism. Then for every $q \in Spec(\hat{A})$, $S^*_k(\hat{M}) \Leftrightarrow P = q^c \in S_k^*(M)$ (for any ideal J we write J^c for $\phi^{-1}(J)$).

Proof. By [5;23.3], $depth_{A_q}$ $(M_p \otimes_{A_p} \hat{A}$

$\mathrm{depth}_{A_p}(M_p) + \mathrm{depth}(\hat{A}_q/pA_p\hat{A}_q)$, since.

$$\bar{\varphi} : A_p \to \hat{A}_q$$
$$\frac{a}{s} \to \frac{\varphi(a)}{\varphi(s)}$$

is a flat homomorphism. Also we have

$$M_p \otimes_{A_p} \hat{A}_q \cong (M \otimes_A A_p) \otimes_{A_p} \hat{A}_q \cong M \otimes_A (A_p \otimes_{A_p} \hat{A}_q)$$
$$\cong M \otimes_A \hat{A}_q \cong M \otimes_A (\hat{A} \otimes_A \hat{A}_q) \cong (M \otimes_A \hat{A}) \otimes_{\hat{A}} \hat{A}_q$$
$$\cong \hat{M} \otimes_{\hat{A}} \hat{A}_q \cong \hat{M}_q.$$

Thus we conclude that

$$\mathrm{depth}_{\hat{A}_q}(\hat{M}_q) = \mathrm{depth}_{A_p} M_p + \mathrm{depth}(\hat{A}_q/pA_p\hat{A}_q).$$

On the other hand, since A is Cohen-Macaulay, $\hat{A}$ is a cohen-Macaulay local ring; whence, by corollary of [5;23.3], $\hat{A}_q/pA_p\hat{A}_q$ is a Cohen-Macaulay ring. But

$$\hat{A}_q/pA_p\hat{A}_q = \hat{A}_q/p\hat{A}_q.$$

Hence

$$\mathrm{depth}(\frac{\hat{A}_q}{pA_p\hat{A}_q}) = \dim(\frac{\hat{A}_q}{p\hat{A}_q}).$$

Moreover, by [5;15.1],

$$\mathrm{ht}\ q = \mathrm{ht}\ p + \dim(\frac{\hat{A}_q}{p\hat{A}_q}).$$

Hence

$$\mathrm{depth}\hat{A}_q(\hat{M}_q) = \mathrm{depth}_{A_p}(M_p) + \mathrm{ht}_q - \mathrm{ht}_p.$$

From which we get, by [5;17.4],

$$\mathrm{dept}\hat{A}_q(\hat{M}_q) + \dim(\frac{\hat{A}}{q}) = \mathrm{depth}_{A_p}(M_p) + \dim\hat{A} - \mathrm{ht}\ p$$
$$= \mathrm{depth}_{A_p}(M_p) + \dim A - \mathrm{ht}\ p$$
$$= \mathrm{depth}_{A_p}(M_p) + \dim(\frac{A}{q})$$

The result now follows.

2. Proposition. With the same assumption as in Proposition 1. Let $p, p' \in \mathrm{Spec}(A)$ be prime ideals such that $p \subseteq p'$ and $p \in S^*_k(M)$. Then $p' \in S^*_k(M)$.

Proof. Since φ: $A \to \hat{A}$ is a faithfully flat homomorphism, there exists $q' \in \mathrm{Spec}(\hat{A})$ for which $(q')^c = p'$ (by [5;7.3]). But φ has the going down property (see[5;9.5]). Hence there is a prime ideal $q \in \mathrm{Spec}(\hat{A})$ such that $q^c = p$ and $q \subseteq q'$. By Proposition 1, this implies that $q \in S^*_k(\hat{M})$. But $\hat{A}$ is a homonorrnic image of a regular local ring (see[5;29.4(ii)]); thus by [3], $S^*_k(\hat{M})$ is a closed subset of $\mathrm{Spec}(\hat{A})$ (note that, every Cohen-Macaulay local ring is biequidimensional ring). This implies that $q' \in S^*_k(\hat{M})$. Again from Proposition 1, this in turn implies that $(q')^c = p' \in S^*_k(M)$ as required.

3. Lemma. (See[4;ch.1, §6, Ex. 1]) Let $R \subseteq T$ be rings and p a minimal prime ideal in R. Then there exists in T a prime ideal contracting to p.

Proof. Let p be a minimal prime ideal of R. Set S=R-p, and

$$K = \{a \mid a \cap S = \phi \ \& \ a \text{ is an ideal of } T\}.$$

Then K have a maximal element which is prime ideal of T. Let q be such prime ideal. Since $(q \cap R) \cap S = \phi$, we have $(q \cap R) \subseteq p$ and consequently $q \cap R = p$.

We now turn to the main theorem of the note.

4. Theorem. For every positive integer k, $S^*_k(M)$ is a closed subset of Spec(A).

Proof: Since $S^*_k(\hat{M})$ is closed in $\mathrm{Spec}(\hat{A})$, there exists an ideal J of $\hat{A}$ such that $V(J) = S^*_k(\hat{M})$. It is enough to show that

$$V(J^c) = S^*_k(M)$$

Let $p \in S^*_k(M)$. Hence there is $q \in S^*_k(\hat{A})$ such that $q^c = p$. Hence $q \in S^*_k(\hat{M})$. Thus $J \subseteq q$; this implies that $J^c \subseteq q^c = p$; i. e., $p \in V(J^c)$.

Now let $p' \in V(J^c)$. φ induces the one-to-one homomorphism

$$\bar{\varphi}: A/J^c \to \hat{A}/J$$

$$a + J^c \to \varphi(a) + J.$$

There is also a minimal prime ideal of J^c as p such that

$$J^c \subseteq p \subseteq p'.$$

Now by Lemma 3, there is q/J in Spec($\hat{A}$/J) such that

$$\bar{\varphi}^{-1}(q/J) = p/J^c$$

Hence $p = q^c$ and $J \subseteq q$. Hence $q \in V(J) = S^*_k(\hat{M})$. It follows from Proposition 1 that $p' \in S^*_k(M)$. By Proposition 2, we conclude that $p' \in S^*_k(M)$.

Hence $V(J^c) = S^*_k(M)$ and $S^*_k(\hat{M})$ is closed as claimed.

5. Corollary. Let B be a homomorphic image of A. Then for every finitely generated B-module N the singular sets $S^*_k(N)$ are closed.

Proof. Let $f: A \in B$ be the relevant ring epimorphism. By [1;5], for every non-negative integer k,

$$S^*_k(N) = \{p \in Spec(B): f^{-1}(p) \in S^*_k(N \mid_A)\}$$

in which $N \mid_A$ is the module N to be considered by restrivtion of scalars by means of f. Since $S^*_k(N \mid_A)$ is a closed subset of Spec (A), and f^*: Spec (B) → Spec (A) is a continuous map, we conclude that $f^{*-1}S^*_k(N \mid_A) = S_k(N)$ is a closed subset of Spec(B).

References

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