

# Characterization of Filters Preserving Reciprocity

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## ABSTRACT

In this paper we characterize the system function of a linear filter that its output will be a reciprocal process whenever its input is a reciprocal one.

## Introduction

Let  $X = \{X(t), -\infty < t < \infty\}$  be a process defined on some complete probability space  $(\Omega, \mathcal{F}, P)$ . The notion of reciprocity was first defined by Jamison [1], and studied in some extent by pasha [2] and [3]. The process  $X$  has reciprocal property on  $(-\infty, \infty)$  if for each  $n \in \mathbb{N}$ , and for each reals  $u < v$ , and for each reals  $t_1, \dots, t_n$  in the complement of interval  $(u, v)$ , and finally for each  $t \in (u, v)$ ,

the conditional distribution of  $X_t$  given  $X_u, X_v, X_{t_1}, \dots, X_{t_n}$  is the same as the conditional distribution of  $X_t$  given  $X_u$  and  $X_v$ .

In [2] a martingale representation of Gaussian stationary reciprocal processes is given. In [3] the notion of reciprocity is generalized. Jamison [1] proved that the covariance function of Gaussian stationary reciprocal processes with zero mean is of the following form

$$C_X(t) = E(X(s)X(t+s)) = be^{-a|t|} \quad t \in \mathbb{R},$$

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for some positive numbers  $a$  and  $b$ . It is clear that  $\sigma^2(X(t)) = b$ .

In this paper we make the following assumptions:

Assumption A: we assume that the process  $X$  satisfies the following conditions:

- (i)  $X$  is Gaussian,
- (ii)  $X$  is stationary,
- (iii) The mean of  $X_t$  is zero
- (iv) The covariance function of  $X_t$  is continuous,
- (v)  $X$  has reciprocal property on  $(-\infty, \infty)$ .

### Linear filters

Let  $X$  be the input of a linear filter with quasi system function  $h$ , i.e.

$$h(t) = 0, \quad t \leq 0.$$

Let  $Y = \{y(t), -\infty < t < \infty\}$  be the output of the system, i.e.

$$Y(t) = \int_0^\infty h(s)X(t-s)ds.$$

It is well known that if the process  $X$  is Gaussian and stationary then the out-put process  $Y$  also is Gaussian and stationary. In the following we want to determine the function  $h$  so that if  $X$  satisfies assumption A, then  $Y$  satisfies the assumption A, specifically it has reciprocal property.

If  $X$  is stationary then the covariance function of  $Y$  is given by

$$\begin{aligned} C_Y(t) &= E(y(t+s)y(s)) \\ &= \int_0^\infty h(s)C_X(s+t)ds \\ &= C_X(t) * h(-s) \end{aligned}$$

where  $*$  stands for the convolution of the function  $C_X(t)$  and  $h_1(t) = h(-t)$ .

We will use the following notions in the sequel:

$$C_X(t) = E(X(t+s)X(s)),$$

$$C_Y(t) = E(Y(t+s)Y(s))$$

$$C_{XY}(t) = E(X(t+s)Y(s))$$

$$S_X(w) = \int_{-\infty}^\infty e^{-itw}C_X(t)dt,$$

$$H(w) = \int_0^\infty e^{-itw}h(t)dt.$$

$S_Y(w), S_{XY}(w)$  will be defined similarly.

**Lemma.** Let  $X$  satisfies assumption A ((i)-(iv)), then

$$S_{XY}(w) = S_X(w)H(-w)$$

$$S_Y(w) = S_{XY}(w)H(w).$$

**Proof.** We have

$$\begin{aligned} C_{XY} &= \int_0^\infty h(s)C_X(s+t)ds \\ &= \int_{-\infty}^\infty h(s)C_X(s+t)ds \\ &= \int_{-\infty}^\infty h(-s)C_X(t-s)ds \\ &= (C_X * h_1)(t) \end{aligned}$$

where  $h_1(t) = h(-t)$ . Therefore by taking the Fourier transform we will have

$$\begin{aligned} S_{XY}(w) &= S_X(w).H_1(w) \\ &= S_X(w)H(-w). \end{aligned}$$

Where  $H_1(w)$  is the fourier transform of  $h_1$ , which is equal to  $H(-w)$ . A similar computation will prove the second equality.

Now we have the following theorem.

**Theorem 1.** Let  $X$  satisfies assumption A ((i)-(v)) and  $C_X(t) = be^{-a|t|}$ . Let  $Y$  be the output of the linear quasi system with system function  $h$ . Then  $Y$  is reciprocal if and only if

$$H(w)H(-w) = \frac{a'b'(a^2 + w^2)}{ab(a'^2 + w^2)}$$

for some positive numbers  $a, b, a', b'$ .

**Proof.** Assume that the input and output of the system satisfies assumption A. From

$$C_X(t) = be^{-a|t|}$$

we get

$$S_X(w) = \frac{2ab}{a^2 + w^2}$$

Similarly, for some  $a' > 0, b' > 0$ , we have

$$C_Y(t) = b'e^{-a'|t|}$$

Therefore

$$S_Y(w) = \frac{2a'b'}{a'^2 + w^2}$$

But, from lemma 1, we have

$$\begin{aligned} S_Y(w) &= S_{XY}(w)H(w) \\ &= S_{XY}(w).H(-w)H(w) \end{aligned}$$

Therefore

$$\frac{2a'b'}{a'^2 + w^2} = \frac{2ab}{a^2 + w^2}H(w)H(-w)$$

From here we get

$$H(w)H(-w) = \frac{a'b'(a^2 + w^2)}{ab(a'^2 + w^2)}$$

Now assume that  $H$  satisfies the above relation and the input process satisfies assumption A ((i)-(v)), we prove that  $Y$  satisfies assumption A ((i)-(v)). The only

property that we have to prove is the reciprocal property of  $Y$ . From lemma 1 and the given condition on  $H$  we have

$$\begin{aligned} S_Y(w) &= S_X(w)H(w).H(-w) \\ &= \frac{2ab}{a^2 + w^2} \cdot \frac{a'b'(a^2 + w^2)}{ab(a'^2 + w^2)} \end{aligned}$$

Thus,

$$S_Y(w) = \frac{2a'b'}{a'^2 + w^2}$$

This is the Fourier transform of a function of the following form

$$C_Y(t) = b'e^{-a'|t|}$$

Now from the Jamison result in [1] we conclude that  $Y$  has reciprocal property.

**Example:** An example of this kind of filters is

$$h(t) = \frac{b'}{b't^2 + b\pi^2}$$

The Fourier transform of  $h$  is

$$H(w) = \sqrt{\frac{b'}{b}}e^{-\pi w\sqrt{\frac{b'}{b}}}$$

Therefore, for any  $a > 0$  we have

$$\begin{aligned} H(w)H(-w) &= \frac{b'}{b} \\ &= \frac{b'a(a^2 + w^2)}{ba(a^2 + w^2)} \end{aligned}$$

This filter will take an input with covariance function

$$C_X(t) = be^{-a|t|}$$

to an output with covariance function

$$C_Y(t) = b'e^{-a'|t|}$$

This filter gives more weight to the most recent input than to the most far inputs.

## References

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