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An Application of Bessel function for Solving Nonlinear Fredholm-Volterra-Hammerstein Integro-differential Equations

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Abstract

In this paper, a collocation method based on the Bessel polynomials is used for the solution of nonlinear Fredholm-Volterra-Hammerstein integro-differential equations (FVHIDEs) under mixed conditions. This method of estimating the solution, transforms the nonlinear (FVHIDEs) to matrix equations with the help of Bessel polynomials of the first kind and collocation points. The matrix equations correspond to a system of nonlinear algebraic equations with the unknown Bessel coefficients. Present results and comparisons demonstrate that our estimate has good degree of accuracy and this method is more valid and useful than other methods.

Introduction

Many problems from physics and engineering and other disciplines lead to linear and nonlinear integral equations. Now, for solution of these equations many analytical and numerical methods have been introduced, but numerical methods are easier than analytical methods and most of the time numerical methods have been used to solve these equations. Ordokhani [1] used Walsh functions operational matrix with Newton-Cotes nodes for solving Fredholm-Hemmerstein integro-differential equations. Authors [2] have solved nonlinear Volterra-Fredholm integro-differential equations by hybrid Legendre polynomials and block-pulse functions. Babolian et al. in [3], obtained solutions of nonlinear VFIDEs by using direct computational method and triangular

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functions. Dehghan and Salehi in [4] have solved the non-linear integro-differential equations based on the meshless method. Arikoglu et al. [5] by using differential transform method obtained numerical solution of integro-differential equations. Yuzbasi et al. [6], Yuzbasi and Sezer [7], Yuzbasi et al. [8] have worked on the Bessel matrix and collocation methods for the numerical solutions of the neutral delay differential equations, the pantograph equations and the Lane-Emden differential equations. Also, readers who are interested in learning more about this topic could refer to [9-15].

Recently, Yazbasi in [16] used Bessel polynomials and Bessel collocation method [8] for solving high-order linear Fredholm-Volterra integro-differential equations.

In this article, using Bessel polynomials and Bessel collocation method we estimate solution of nonlinear (FVIDEs) to form:

$$\sum_{k=0}^{n} p_k(x)y^{(k)}(x) = g(x) + \lambda_1 \int_a^b k_1(x,t)\psi_1(t,y(t))\,dt + \lambda_2 \int_a^x k_2(x,t)\psi_2(t,y(t))\,dt, \quad (1)$$

$$0 \le a \le x, t \le b,$$

under the mixed conditions

$$\sum_{k=0}^{n-1}[a_{jk}\, y^{(k)}(a) + b_{jk} y^{(k)}(b)] = \lambda_j, \qquad j = 0,1,\cdots,n-1, \quad (2)$$

where y(x) is an unknown function, the known functions are $p_k(x)$, k=0,1, ... ,n, g(x), $k_1(x,t), k_2(x,t), \psi_1(t,y(t))$ and $\psi_2(t,y(t))$. Also, $a_{jk}, b_{jk}, \lambda_1, \lambda_2$ and λ_j are real or complex constants.

Introductory properties of Bessel and Taylor polynomials

1. Bessel polynomials of first kind

The m-th degree truncated Bessel polynomials of first kind are defined by [16]

$$J_m(x) = \sum_{k=0}^{[\frac{N-m}{2}]} \frac{(-1)^k}{k!(k+m)!} (\frac{x}{2})^{2k+m}, \qquad 0 \le x < \infty, \qquad m \in \mathbb{N}, \quad (3)$$

where N is chosen a positive integer so that $N \ge n$ and $m = 0,1,\cdots,N$.

we can transform the Bessel polynomials of first kind to N-th degree Taylor basis functions. In matrix form as

$$J(x) = DX(x), \tag{4}$$

where

$$J(x) = [J_0(x), J_1(x), \cdots, J_N(x)]^T, \qquad X(x) = [1, x, x^2, \cdots, x^N]^T. \tag{5}$$

If N is odd

$$D = \begin{bmatrix} \frac{1}{0!\,0!\,2^0} & 0 & \frac{-1}{1!\,1!\,2^2} & \cdots & \frac{(-1)^{\frac{N-1}{2}}}{(\frac{N-1}{2})!\,(\frac{N-1}{2})!\,2^{N-1}} & 0 \\ 0 & \frac{1}{0!\,1!\,2^1} & 0 & \cdots & 0 & \frac{(-1)^{\frac{N-1}{2}}}{(\frac{N-1}{2})!\,(\frac{N-1}{2})!\,2^{N}} \\ 0 & 0 & \frac{1}{0!\,2!\,2^2} & \cdots & \frac{(-1)^{\frac{N-3}{2}}}{(\frac{N-3}{2})!\,(\frac{N+1}{2})!\,2^{N-1}} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{(-1)^{\frac{N-1}{2}}}{0!\,(N-1)!\,2^{N-1}} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{0!\,N!\,2^N} \end{bmatrix}_{(N+1)\times(N+1)}.$$

If N is even

$$D = \begin{bmatrix} \frac{1}{0!\,0!\,2^0} & 0 & \frac{-1}{1!\,1!\,2^2} & \cdots & 0 & \frac{(-1)^{\frac{N}{2}}}{\left(\frac{N}{2}\right)!\left(\frac{N}{2}\right)!\,2^{N}} \\ 0 & \frac{1}{0!\,1!\,2^1} & 0 & \cdots & \frac{(-1)^{\frac{N-2}{2}}}{\left(\frac{N-2}{2}\right)!\left(\frac{N}{2}\right)!\,2^{N-1}} & 0 \\ 0 & 0 & \frac{1}{0!\,2!\,2^2} & \cdots & 0 & \frac{(-1)^{\frac{N-2}{2}}}{\left(\frac{N-2}{2}\right)!\left(\frac{N+2}{2}\right)!\,2^{N}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{0!\,(N-1)!\,2^{N-1}} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{0!\,N!\,2^N} \end{bmatrix}_{(N+1)\times(N+1)}.$$

2. Taylor polynomials operational matrix of integration

We considere the vector of Taylor polynomials X(x) in (5) with its integration obtained as [17]

$$\int_0^x X(t)dt \simeq LX(x), \tag{6}$$

so that dimension of L is (N + 1) × (N + 1) and

$$L=\begin{bmatrix}0 & 1 & 0 & \cdots & 0\\ 0 & 0 & \frac{1}{2} & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & \frac{1}{N}\\ 0 & 0 & 0 & \cdots & 0\end{bmatrix}, \tag{7}$$

where L is operational matrix of integral for Taylor polynomials. We also present dual operational matrix of $X(x)$ with taking the integration of the cross product of two vector function of Taylor polynomials as [17]

$$H=\int_a^b X(t)X^T(t)dt, \qquad H=[h_{ij}], \quad i,j=0,1,\cdots,N, \tag{8}$$

Where

$$h_{ij}=\frac{b^{i+j+1}-a^{i+j+1}}{i+j+1}, \qquad i,j=0,1,\cdots,N. \tag{9}$$

Fundamental relations

1. Matrix relation for the Fredholm integral part

In this section we can approximate the kernel function $k_1(x,t)$ by the truncated Maclaurin series and truncated Bessel series [16], respectively

$$\begin{cases} k_1(x,t)=\sum_{m=0}^{N}\sum_{n=0}^{N} {}_tk_{mn}^1\, x^m t^n,\\ k_1(x,t)=\sum_{m=0}^{N}\sum_{n=0}^{N} {}_bk_{mn}^1\, J_m(x)J_n(t),\end{cases} \tag{10}$$

where

$${}_tk_{mn}^1=\frac{1}{m!\,n!}\frac{\partial^{m+n}k_1(0,0)}{\partial x^m\partial t^n}, \qquad m,n=0,1,\cdots,N.$$

We can write matrix forms of Eq. (10) as

$$k_1(x,t)=X^T(x)k_t^1X(t), \qquad k_t^1=[\,{}_tk_{mn}^1], \qquad m,n=0,1,\cdots,N, \tag{11}$$

$$k_1(x,t)=J^T(x)k_b^1J(t), \qquad k_b^1=[\,{}_bk_{mn}^1], \qquad m,n=0,1,\cdots,N, \tag{12}$$

By substituting Eq. (4) in Eq. (12) and putting equal to Eq. (11) we obtain:

$$k_t^1 = D^T\, k_b^1 D, \qquad k_b^1 = (D^T)^{-1} k_t^1 (D)^{-1}. \tag{13}$$

Now, for solving these equations, we need to define $Z_1(t)$ and $Z_2(t)$ as

$$Z_1(t) = \psi_1(t, y(t)), \tag{14}$$
$$Z_2(t) = \psi_2(t, y(t)),$$

and approximate them by Bessel polynomials of first kind and using Eq. (4)

$$Z_1(t) \simeq J^T(t) A_1 = X^T(t) D^T A_1, \tag{15}$$
$$Z_2(t) \simeq J^T(t) A_2 = X^T(t) D^T A_2,$$

where

$$A_1 = [a_{10}, a_{11}, \cdots, a_{1N}]^T, \qquad A_2 = [a_{20}, a_{21}, \cdots, a_{2N}]^T.$$

By substituting the matrix forms of Eqs. (12) and (15) in Fredholm integral part of Eq. (1) we get

$$\int_a^b k_1\,(x,t)\psi_1(t, y(t)) \simeq \int_a^b J^T\,(x) k_b^1\, J(t) J^T(t) A_1 dt = J^T(x) k_b^1\, Q_1 A_1, \tag{16}$$

so that

$$Q_1 = \int_a^b J(t)\, J^T(t) dt \simeq \int_a^b D X(t) X^T(t) D^T\, dt = D H_1 D^T,$$

where H_1, the integration of dual operational matrix of Taylor polynomials, is defined in (8). Finally, by substituting Eq. (4) in Eq. (16) we have matrix form of Fredholm part

$$\int_a^b k_1\,(x,t)\psi_1(t, y(t)) \simeq X^T(x) D^T k_b^1\, Q_1 A_1. \tag{17}$$

2. Matrix relation for the Volterra integral part

We can write kernel function $k_2(x,t)$ such as $k_1(x,t)$ and approximate it by truncated Maclaurin series and truncated Bessel series [16]

$$k_2(x,t) = \sum_{m=0}^{N} \sum_{n=0}^{N}\ {}_t k_{mn}^2\, x^m t^n, \tag{18}$$

$$k_2(x,t) = \sum_{m=0}^{N}\sum_{n=0}^{N} {}_b k_{mn}^2 J_m(x) J_n(t),$$

where

$${}_t k_{mn}^2 = \frac{1}{m!\,n!}\frac{\partial^{m+n} k_2(0,0)}{\partial x^m \partial t^n}, \qquad m,n = 0,1,\cdots,N.$$

Matrix form as

$$k_2(x,t) = X^T(x) k_t^2 X(t), \qquad k_t^2 = [\,{}_t k_{mn}^2], \qquad m,n = 0,1,\cdots,N, \tag{19}$$

$$k_2(x,t) = J^T(x) k_b^2 J(t), \qquad k_b^2 = [\,{}_b k_{mn}^2], \qquad m,n = 0,1,\cdots,N. \tag{20}$$

By substituting Eq. (4) in Eq. (20) and putting equal to Eq. (19) we obtain

$$k_t^2 = D^T\, k_b^2 D, \qquad k_b^2 = (D^T)^{-1} k_t^2 (D)^{-1}. \tag{21}$$

By substituting the matrix form of Eqs. (15) and (20) in Volterra integral part of Eq. (1) we have

$$\int_a^x k_2(x,t)\psi_2(t,y(t)) \simeq \int_a^x J^T(x) k_b^2 J(t) J^T(t) A_2 dt = J^T(x) k_b^2\, Q_2(x) A_2, \tag{22}$$

so that

$$Q_2(x) = \int_a^x J(t)\, J^T(t) dt \simeq \int_a^x D X(t) X^T(t) D^T\, dt = D H_2(x) D^T,$$

where $H_2(x)$, the integration of dual operational matrix of Taylor polynomials, is defined as

$$H_2(x) = \int_a^x X(t) X^T(t) dt = [h'_{ij}(x)], \qquad i,j = 0,1,\cdots,N,$$

$$h'_{ij}(x) = \frac{x^{i+j+1} - a^{i+j+1}}{i+j+1}, \qquad i,j = 0,1,\cdots,N.$$

By substituting Eq. (4) in Eq. (22) we have matrix form of Volterra part

$$\int_a^x k_2(x,t)\psi_2(t,y(t)) \simeq X^T(x) M H_2(x) D^T A_2, \qquad M = D^T k_b^2 D. \tag{23}$$

3. Method of solution

To solve Eq. (1) with conditions in Eq. (2), we assume

$$y^{(n)}(x) \simeq \sum_{i=0}^{N} a_i J_i(x) = A^T J(x) = J^T(x) A, \tag{24}$$

where

$$A = [a_0, a_1, \cdots, a_N]^T.$$

By using Eqs. (4), (6) and (24) we have

$$y^{(n-1)}(x) \simeq A^T \int_0^x J(t)dt + \lambda_{n-1},$$

$$\simeq A^T \int_0^x DX(t)dt + \lambda_{n-1} = A^T DLX(x) + \lambda_{n-1}$$

and

$$y^{(n-2)}(x) \simeq A^T DL^2 X(x) + \lambda_{n-1} x + \lambda_{n-2},$$

similarly for $y(x)$ we obtain

$$y(x) \simeq A^T DL^n X(x) + \sum_{j=0}^{n-1} \frac{1}{(n-1-j)!} \lambda_{n-1-j} x^{n-1-j}. \tag{25}$$

Now, by substituting Eqs. (17), (23) and (25) in Eq. (1) we get

$$\sum_{k=0}^{n} p_k(x)\left(A^T DL^{n-k} X(x) + \sum_{j=0}^{n-k-1} \frac{1}{(n-1-j)!} \lambda_{n-1-j} x^{n-1-j}\right) \tag{26}$$

$$= g(x) + \lambda_1 X^T(x) D^T k_b^1\, Q_1 A_1 + \lambda_2 X^T(x) M H_2(x) D^T A_2\ .$$

We can expand $x^i, i = 0,1, \cdots, n-1$, with Taylor bases to get

$$x^i = e_i X(x),$$
$$e_i = (\underbrace{0,0,\cdots,0,}_{i-1} 1, \underbrace{0,\cdots,0}_{N-i})^T, \qquad i = 0,1,\cdots,n-1, \qquad n \le N. \tag{27}$$

By using Eq. (27) and substituting it in Eqs. (25) and (26) we have, respectively

$$y(x) \simeq A^T DL^n X(x) + \sum_{j=0}^{n-1} \frac{1}{(n-1-j)!} \lambda_{n-1-j} e_{n-1-j}\, X(x), \tag{28}$$

and

$$\sum_{k=0}^{n} p_k(x)\left(A^T DL^{n-k} X(x) + \sum_{j=0}^{n-k-1} \frac{1}{(n-1-j)!} \lambda_{n-1-j} e_{n-1-j}\, X(x)\right) \tag{29}$$

$$= g(x) + \lambda_1 X^T(x) D^T k_b^1\, Q_1 A_1 + \lambda_2 X^T(x) M H_2(x) D^T A_2.$$

By using Eq. (28) and substituting this equation in Eq. (15) we obtain

$$\psi_1(x, A^T D L^n X(x) + \sum_{j=0}^{n-1} \frac{1}{(n-1-j)!} \lambda_{n-1-j} e_{n-1-j}\, X(x)) = X^T(x) D^T A_1, \tag{30}$$

$$\psi_2(x, A^T D L^n X(x) + \sum_{j=0}^{n-1} \frac{1}{(n-1-j)!} \lambda_{n-1-j} e_{n-1-j}\, X(x)) = X^T(x) D^T A_2. \tag{31}$$

Also, from Eqs. (29), (30) and (31) and collocation points [16] defined by

$$x_i = a + \frac{b-a}{N} i, \qquad i = 0,1,\cdots,N.$$

We have

$$\begin{cases} \sum_{k=0}^{n} p_k\,(x_i)(\, A^T D L^{n-k} X(x_i) + \sum_{j=0}^{n-k-1} \frac{1}{(n-1-j)!} \lambda_{n-1-j} e_{n-1-j}\, X(x_i)) \\ \qquad = g(x_i) + \lambda_1 X^T(x_i) D^T k_b^1\, Q_1 A_1 + \lambda_2 X^T(x_i) M H_2(x_i) D^T A_2, \\ \psi_1(x_i, A^T D L^n X(x_i) + \sum_{j=0}^{n-1} \frac{1}{(n-1-j)!} \lambda_{n-1-j} e_{n-1-j}\, X(x_i)) = X^T(x_i) D^T A_1, \\ \psi_2(x_i, A^T D L^n X(x_i) + \sum_{j=0}^{n-1} \frac{1}{(n-1-j)!} \lambda_{n-1-j} e_{n-1-j}\, X(x_i)) = X^T(x_i) D^T A_2, \end{cases} \tag{32}$$

where $\mathrm{i} = 0,1,\cdots,\mathrm{N}$. Or briefly the fundamental matrix system is

$$\begin{cases} \sum_{k=0}^{n} P_k\,(\, A^T D L^{n-k} X^T + \sum_{j=0}^{n-k-1} \frac{1}{(n-1-j)!} \lambda_{n-1-j} e_{n-1-j}\, X^T) \\ \qquad = G + \lambda_1 X D^T k_b^1\, Q_1 A_1 + \lambda_2 \bar{X} \bar{M} \bar{H} \bar{D}\, A_2, \\ \psi_1(x_i, A^T D L^n X^T + \sum_{j=0}^{n-1} \frac{1}{(n-1-j)!} \lambda_{n-1-j} e_{n-1-j}\, X^T) = X D^T A_1, \\ \psi_2(x_i, A^T D L^n X^T + \sum_{j=0}^{n-1} \frac{1}{(n-1-j)!} \lambda_{n-1-j} e_{n-1-j}\, X^T) = X D^T A_2, \end{cases} \tag{33}$$

where $\mathrm{i} = 0,1,\cdots,\mathrm{N}$ and

$$P_k = \begin{bmatrix} p_k(x_0) & 0 & \cdots & 0 \\ 0 & p_k(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_k(x_N) \end{bmatrix}, \quad G = \begin{bmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ g(x_N) \end{bmatrix}, \quad X = \begin{bmatrix} X^T(x_0) \\ X^T(x_1) \\ \vdots \\ X^T(x_N) \end{bmatrix},$$

$$\bar{M} = \begin{bmatrix} M & 0 & \cdots & 0 \\ 0 & M & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M \end{bmatrix}, \quad \bar{X} = \begin{bmatrix} X^T(x_0) & 0 & \cdots & 0 \\ 0 & X^T(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X^T(x_N) \end{bmatrix},$$

$$\overline{D}=\begin{bmatrix} D^T \\ D^T \\ \vdots \\ D^T \end{bmatrix} \quad and \quad \overline{H}=\begin{bmatrix} H_2(x_0) & 0 & \cdots & 0 \\ 0 & H_2(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & H_2(x_N) \end{bmatrix}.$$

We can obtain A_1, A_2 and A from system of Eq. (33) and with substituting A in Eq. (28) ultimately, we get approximate solution of Eq. (1).

Illustrative examples

In this section, we report the results of approximation solution to some examples that were given various papers. In addition, we have expressed absolute error functions which are defined as $|y(x)-y_N(x)|$, where $y(x)$ is the exact solution of Eq. (1) and $y_N(x)$ is the approximate of $y(x)$. All the examples were performed on the computer by using a program written in MATLAB.

Example 1. Let us first consider the nonlinear FHIDE [18],

$$y''(x)+xy'(x)-xy(x)=e^x-sin(x)+\int_0^1 sin(x)e^{-2t}\,y^2(t)dt,\ \ 0\le x,t\le 1, \tag{34}$$

with conditions y(0) = 1, $y'(0)=1$ and the exact solution to Eq. (34) is y(x) = exp(x). Now, we obtain approximate solutions of this example for N=2, 4, 6, 8 by Bessel polynomials. where $p_0(x)=-x,\ p_1(x)=x,\ p_2(x)=1,\ k_1(x,t)=\sin(x)e^{-2t},\ \lambda_1=1$ $g(x)=e^x-\sin(x)\ .\ and$ Also, the set of collocation points for N = 2 is

$$\left\{x_0=0,x_1=\frac{1}{2},x_2=1\right\}$$

so that

$$X=\begin{bmatrix} 1 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{4} \\ 1 & 1 & 1 \end{bmatrix},\qquad D=\begin{bmatrix} 1 & 0 & \frac{-1}{4} \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{8} \end{bmatrix},\qquad L=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix},$$

$$P_0=\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{-1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix},\qquad P_1=\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix},\qquad P_2=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$k_{mn}^1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \quad k_b^1 = \begin{bmatrix} 0 & 0 & 0 \\ 2 & -8 & 36 \\ 0 & 0 & 0 \end{bmatrix}, \quad H_1 = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix},$$

$$Q = \begin{bmatrix} 8.458 \times 10^{-1} & 2.187 \times 10^{-1} & 3.541 \times 10^{-2} \\ 2.187 \times 10^{-1} & 8.333 \times 10^{-2} & 1.562 \times 10^{-2} \\ 3.541 \times 10^{-2} & 1.562 \times 10^{-2} & 3.125 \times 10^{-3} \end{bmatrix}, \quad G = \begin{bmatrix} 1 \\ 1.1693 \\ 1.8768 \end{bmatrix}.$$

Hence, by using system (32) and matrices obtained above, we have

$$\begin{cases} \sum_{k=0}^{2} P_k \left(A^T D L^{2-k} X^T + \sum_{j=0}^{1-k} \frac{1}{(1-j)!} \lambda_{1-j} e_{1-j}\, X^T \right) = G + \lambda_1 X D^T k_b^1\, Q_1 A_1, \\ \psi_1\left(x_i, A^T D L^2 X^T + \sum_{j=0}^{1} \frac{1}{(1-j)!} \lambda_{1-j} e_{1-j}\, X^T\right) = X D^T A_1, \end{cases}$$

we obtain Bessel coefficient matrix as

$$A = [1 \quad 4.1954 \quad 4.0252]^T.$$

Ultimately, by substituting A in Eq. (28) for N = 2 and n = 2 we have approximate solution of Eq. (34)

$$y_2(x) = 1 + x + 0.5x^2.$$

Similarly for N = 4, 6, 8 we have

$$y_4(x) = 1 + x + 0.5x^2 + 0.199308331x^3 + 0.0479812523x^4,$$

$$y_6(x) = 1 + x + 0.5x^2 + 0.169341666\mathrm{x}^3 + 0.0416791666\mathrm{x}^4 + (0.7754895833\ 10^{-2})\mathrm{x}^5 + (0.15533159722 \times 10^{-2})\mathrm{x}^6$$

and

$\mathrm{y}_8(\mathrm{x}) = 1 + \mathrm{x} + 0.5\mathrm{x}^2 + 0.166791666\mathrm{x}^3 + 0.041659375\mathrm{x}^4 + (0.834427083 \times$ 10−2x5+(0.128079861×10−2)x6+(0.42292906746×10−3)x7−(0.2513107026×1 0−3)x8.

The absolute error values are given for different values of N in Table 1.

Table1. Absolute errors and CPU times of Example 1

x	Present method N=4	N=6	N=8	Method of [18] n=5, m=5
0.0	0	0	0	0
0.2	2.68×10^{-4}	2.12×10^{-5}	9.87×10^{-7}	4.00×10^{-7}
0.4	2.15×10^{-3}	1.65×10^{-4}	6.77×10^{-6}	8.10×10^{-6}
0.6	7.15×10^{-3}	5.36×10^{-4}	2.34×10^{-5}	7.73×10^{-5}
0.8	1.61×10^{-2}	1.18×10^{-3}	3.66×10^{-5}	4.24×10^{-4}
1.0	2.90×10^{-2}	2.04×10^{-3}	3.40×10^{-5}	1.64×10^{-3}
CPU	3.08×10^{-2}	5.84×10^{-2}	4.78×10^{-2}	-

Example 2. Now consider the nonlinear VHIDE [2],

$$y'(x) = -2sin(x) - \frac{1}{3}cos(x) - \frac{2}{3}cos(2x) + \int_0^x cos(x-t)y^2(t)dt,$$

$$0 \le x \le 1,$$

with condition y(0) = 1. The exact solution to this equation is y(x) = cos(x)-sin(x). The values obtained in Table 2 show that if the accuracy increases, N will increase .

Table2. Absolute errors and CPU times of Example 2

x	Present method N=4	N=6	N=8	Method of [2] n=8, m=8
0.0	0	0	0	1.00×10^{-6}
0.1	1.07×10^{-5}	1.75×10^{-7}	4.52×10^{-9}	1.60×10^{-5}
0.2	3.26×10^{-5}	5.68×10^{-7}	2.13×10^{-8}	2.56×10^{-4}
0.3	7.65×10^{-5}	1.23×10^{-6}	4.68×10^{-8}	8.40×10^{-5}
0.4	1.98×10^{-4}	2.63×10^{-6}	6.94×10^{-8}	9.43×10^{-4}
0.5	5.09×10^{-4}	6.29×10^{-6}	1.89×10^{-8}	1.20×10^{-5}
0.6	1.19×10^{-3}	1.61×10^{-5}	3.93×10^{-7}	2.76×10^{-4}
0.7	2.50×10^{-3}	4.10×10^{-5}	2.07×10^{-6}	4.70×10^{-5}
0.8	4.82×10^{-3}	9.74×10^{-5}	7.43×10^{-6}	1.10×10^{-5}
0.9	8.60×10^{-3}	2.14×10^{-4}	2.19×10^{-5}	7.80×10^{-5}
1.0	1.44×10^{-2}	4.39×10^{-4}	5.72×10^{-5}	8.15×10^{-4}
CPU	2.37×10^{-2}	3.08×10^{-2}	4.07×10^{-2}	-

Example 3**.** Consider the nonlinear FVHIDE [4],

$$y'(x) = 1 - \frac{x}{2} + \frac{x}{2} exp(-x^2) + \int_0^x xt\, exp(-y^2(t))dt, \quad 0 \le x \le 1,$$

with condition y(0) = 0. The exact solution to this example is y(x) = x. The maximum absolute errors and CPU times are shown in Table 3 and the absolute error for various values N are shown in Table 4.

Table3. Maximum absolute errors and CPU times of Example 3

N=5	Present method	Method of [4]
CPU	2.36×10^{-2}	0.39
Maximum absolute error	5.81×10^{-5}	2.02×10^{-4}

Table 4: Absolute errors and CPU times of Example 3

x	Present method		
	N=2	N=3	N=5
0.0	0	0	0
0.1	4.88×10^{-5}	1.07×10^{-7}	3.91×10^{-8}
0.2	1.95×10^{-4}	7.23×10^{-7}	6.14×10^{-8}
0.3	4.39×10^{-4}	2.30×10^{-6}	9.05×10^{-9}
0.4	7.80×10^{-4}	5.29×10^{-6}	2.05×10^{-7}
0.5	1.22×10^{-3}	1.01×10^{-5}	8.74×10^{-7}
0.6	1.75×10^{-3}	1.73×10^{-5}	2.67×10^{-6}
0.7	2.39×10^{-3}	2.72×10^{-5}	6.85×10^{-6}
0.8	3.12×10^{-3}	4.03×10^{-5}	1.53×10^{-5}
0.9	3.95×10^{-3}	5.71×10^{-5}	3.11×10^{-5}
1.0	4.88×10^{-3}	7.80×10^{-5}	5.81×10^{-5}
CPU	1.26×10^{-2}	2.33×10^{-2}	2.36×10^{-2}

Example 4. Consider the nonlinear FVHIDE [2,3,19],

$$y'(x)+y(x)=2x+x^2+\frac{1}{10}x^6-\frac{1}{32}+\frac{1}{4}\int_0^1 ty^3(t)dt-\frac{1}{2}\int_0^x xy^2(t)dt, \quad 0\le x\le 1,$$

with condition y(0) = 0. The exact solution to this example is $y(x)=x^2$. The computational results of absolute error for N = 4 and N = 6 with the result of other methods are given in Table 5.

Table 5. Absolute errors and CPU times of Example 4

x	Present method N=4	N=6	Method of [2] n=8,m=8	Method of [3]	Method of [19] M=16
0.0	0	0	0	0	0
0.1	3.34×10^{-5}	5.66×10^{-8}	2.18×10^{-3}	3.10×10^{-5}	1.66×10^{-4}
0.2	6.37×10^{-5}	1.01×10^{-7}	1.46×10^{-3}	7.50×10^{-5}	2.54×10^{-4}
0.3	9.12×10^{-5}	1.33×10^{-7}	1.67×10^{-3}	1.71×10^{-4}	2.62×10^{-4}
0.4	1.16×10^{-4}	1.60×10^{-7}	7.23×10^{-3}	9.40×10^{-5}	1.91×10^{-4}
0.5	1.38×10^{-4}	2.13×10^{-7}	2.28×10^{-4}	1.60×10^{-2}	4.10×10^{-5}
0.6	1.60×10^{-4}	3.71×10^{-7}	1.14×10^{-2}	5.02×10^{-4}	2.02×10^{-4}
0.7	1.80×10^{-4}	8.80×10^{-7}	4.51×10^{-3}	5.83×10^{-4}	2.83×10^{-4}
0.8	1.99×10^{-4}	1.90×10^{-7}	4.87×10^{-3}	3.74×10^{-4}	2.83×10^{-4}
0.9	2.19×10^{-4}	4.25×10^{-7}	1.66×10^{-2}	4.70×10^{-5}	2.02×10^{-4}
1.0	2.38×10^{-4}	8.83×10^{-7}	3.09×10^{-2}	1.40×10^{-5}	3.70×10^{-5}
CPU	8.86×10^{-2}	9.01×10^{-2}	-	-	-

Conclusion

In this paper, we have solved nonlinear FVHIDEs by Bessel polynomials of the first kind and collocation method. One significant advantage of this method is that by increasing value of N, approximate solution is convergent and the accuracy increases sufficiently. This method can produce sparse matrix, and this is one of the major reasons for its high accuracy, and, as noticed earlier, the results of the proposed method are more accurate than the results of Legendre-hybrid polynomials.

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